σ-SET THEORY: INTRODUCTION TO THE CONCEPTS OF σ-ANTIELEMENT, σ-ANTISET AND INTEGER SPACE.

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ABSTRACT. In this paper we develop a theory called σ-Set Theory, in which we present an axiom system developed from the study of Set Theories of Zermelo-Franquel, Neumann-Bernays-Godel and Morse-Kelley. In σ-Set Theory, we present the proper existence of objects called σ-antielement, σ-antiset, natural numbers, antinatural numbers and generated σ-set by two σ-sets, from which we obtain, among other things, a commutative non-associative algebraic structure called Integer Space 3^X, which corresponds to the algebraic completion of 2^X.

1. INTRODUCTION

The σ-Set Theory comes from the study of multivalued operators $T : X \to 2^X$ defined in a Banach Space X with images in 2^X. From here it seemed interesting to investigate the properties of this type of operators in the case that 2^X is algebraically complete, i.e., when it is added the inverse elements for the union.

On the other hand, since the discovery of matter-antimatter duality, many mathematicians have been interested in translating this physical phenomenon into Set-Theoretic language. For this reason, we consider this goal in our work. Thus, we have the following scheme:

Physic	$\to$	σ-Set Theory
particle	$\to$	σ-element
antiparticle	$\to$	σ-antielement
matter	$\to$	σ-set
antimatter	$\to$	σ-antiset
ξ energy	$\to$	$\emptyset$ emptyset.

1991 *Mathematics Subject Classification.* 03E47.

During the development of this work, we had the initial support of Victor Cardenas Vera, Diego Lobos Maturana, Mariela Carvacho Bustamente and Hector Carrasco Altamirano, which motivated us to move forward. We also received help from Rafael Espinola Garcia, Carlos Jiménez Gómez, Carlos Hernandez Linares, Andreea Sarafoleanu and Francisco Canto Martin in the correction and translation of the manuscript. Without this help we would not been able to finish our work.

Respect to the model used to build the axiomatic system of σ-Set Theory, we observe that we first use the axioms of ZF Set Theory. However, after many tries to obtain a consistent axiom system, we realized that this model is not sufficient to given a proper existence to the concept of σ-antielement and σ-antiset. For this reason, we use the axioms system of NBG Set Theory and MK Set Theory. Therefore, the generic object of σ-Set Theory is called σ-class.

In the present article, we develop and individual analysis of the axioms and so we present the results that we consider necessary for the creation of the σ-antielements and σ-antisets. Concerning the axiom system of σ-Set Theory we will have that: The axioms 3.1 (Empty σ-Set), 3.2 (Extensionality), 3.3 (Creation of σ-Class), 3.4 (Replacement) and 3.14 (Power σ-Set) are analogous to some axioms of ST. Axioms 3.5 (Pairs) and 3.15 (Fusion) are modifications of the axioms of "pairs" and "union". Axiom 3.6 (Weak Regularity) is a modification of the axiom of Regularity. Axiom 3.7 (non ϵ-Bounded σ-Set) is a extension of the axiom of infinity. Axioms 3.8 (Weak Choice), 3.9 (ϵ-Linear σ-Set) and 3.10 (One and One* σ-Sets) are necessary to define the concept of σ-antielement. With axioms 3.11 (Completeness A)and 3.12 (Completeness B) we give the rules of construction of pairs, which will be used to decide when a σ-set has a σ-antielement. Axiom 3.13 (Exclusion) will be used to define the fusion of σ-sets, which will help us to define the σ-antiset. Finally, axiom 3.16 (Generated σ-Set) guarantees the existence of generated σ-set by two σ-sets from which we obtain the Integer Space 3^X, that in some sense will be more bigger than 2^X. Therefore, we will see that if we consider this axiom system, then we can build the following concepts:

- **σ-Antielement**: x^* is a σ-antielement of x if and only if
$$\{x\} \cup \{x^*\} = \emptyset.$$
- **σ-Antiset**: $X^\star$ is a σ-antiset of X if and only if
$$X \cup X^\star = \emptyset.$$
- **Natural numbers**: $IN = \{1, 2, 3, 4, \ldots\}$, where
$$1 = \{\alpha\},\ 2 = \{\alpha, 1\},\ 3 = \{\alpha, 1, 2\},\ etc...$$
- **Θ-Natural numbers**: $IN_\Theta = \{1_\Theta, 2_\Theta, 3_\Theta, 4_\Theta, \ldots\}$, where
$$1_\Theta = \{\emptyset\},\ 2_\Theta = \{\emptyset, 1_\Theta\},\ 3_\Theta = \{\emptyset, 1_\Theta, 2_\Theta\},\ etc...$$
- **Antinatural numbers**: $IN^\star = \{1^*, 2^*, 3^*, 4^*, \ldots\}$, where
$$1^* = \{\beta\},\ 2^* = \{\beta, 1^*\},\ 3^* = \{\beta, 1^*, 2^*\},\ etc...$$
- **Power σ-set**: $2^X = \{x : x \subseteq X\}$.
- **Generated space**: $\langle 2^X, 2^Y\rangle = \{x \cup y : x \in 2^X \wedge y \in 2^Y\}$.

- **Integer space**: $3^X = \langle 2^X, 2^{X^\star} \rangle = \{x \cup y : x \in 2^X \wedge y \in 2^{X^\star}\}$.

The formal definitions of these concepts we will see in the analysis of axioms, in particular, when we present the axioms 3.8, 3.9 and 3.10 we will see the relation of σ-sets α and β. In Section 4 (Final Comments) we will see some basic properties of the natural σ-sets, i.e., natural numbers, antinatural numbers and Θ-natural numbers. Also we present some conjectures which are particularly related with **loops** and **lattices**. Finally, in this Section we present a summary of the axiomatic system of σ-Set Theory.

With respect to the presentation of this work we must stand out that we have continued the introduction to the Set Theory that is presented in the books by K. Devlin [2], K. Hrbacek and T. Jech [4], C. Ivorra [5] and T. Jech [6] in order to show a formal scheme similar to that presented on Set Theory.

Finally we want to emphasize that the idea of introducing the inverse element as it is known, is not original. In particular, we have seen that the idea of antiset has already been considered in Mathematics (see, W.D. Blizard [1] (Notes), Fishburn and I. H. La Valle [3] and V. Pratt [7]), for this reason we hope this work will be well received.

2. Introduction to σ-Set Theory

The language of the σ-Set Theory, σ-LST, is a formal language with predicates "$=$" (equality) and "$\in$" (is a σ-element of), logical symbols "$\wedge$" (and), "$\vee$" (or), "$\neg$" (not), "$\exists$" (there exists) and "$\forall$" (for all), variables $\hat{X}, \hat{Y}, \hat{Z}, ...$ and (for convenience) brackets $(\cdot)$, $[\cdot]$. We use the following abbreviations

- $(\forall \hat{Y} \in \hat{X})(\Psi) := (\forall \hat{Y})(\hat{Y} \in \hat{X} \to \Psi)$;
- $(\exists \hat{Y} \in \hat{X})(\Psi) := (\exists \hat{Y})(\hat{Y} \in \hat{X} \wedge \Psi)$;
- $(\exists! \hat{Y} \in \hat{X})(\Psi) := (\exists! \hat{Y})(\hat{Y} \in \hat{X} \wedge \Psi)$;
- $cto\hat{X} := (\exists \hat{Y})(\hat{X} \in \hat{Y})$.

To refer to a generic object we will use the word σ-class, i.e., agree that a formula of type $(\forall \hat{X})(\Phi)$ is read "All σ-class $\hat{X}$ satisfies Φ". Also, the formula, $cto\hat{X}$, is read "$\hat{X}$ is a σ-set". So, we defined the σ-sets as the σ-classes that belong to at least one other σ-class.

The difference between σ-class and σ-set, will be see reflected in the axiom system. A σ-class which is not a σ-set is called a proper σ-class.

We use Roman letters x, y, z, X, Y, Z, etc. to denote σ-sets, i.e.,

- $(\forall x)(\Phi) := (\forall X)(\Phi) := (\forall \hat{X})(cto\hat{X} \to \Phi)$;
- $(\exists x)(\Phi) := (\exists X)(\Phi) := (\exists \hat{X})(cto\hat{X} \wedge \Phi)$;
- $(\exists! x)(\Phi) := (\exists! X)(\Phi) := (\exists! \hat{X})(cto\hat{X} \wedge \Phi)$.

Also we use the following abbreviations

- $(\hat{X} \notin \hat{Y}) := \neg(\hat{X} \in \hat{Y})$;
- $(\hat{X} \neq \hat{Y}) := \neg(\hat{X} = \hat{Y})$;
- $(\Phi \underline{\vee} \Psi) := (\Phi \wedge \neg\Psi) \vee (\neg\Phi \wedge \Psi)$;

- $(\hat{Y} \subseteq \hat{Z}) := (\forall \hat{X})(\hat{X} \in \hat{Y} \to \hat{X} \in \hat{Z})$;
- $(\hat{Y} \subset \hat{Z}) := (\hat{Y} \subseteq \hat{Z} \wedge \hat{Y} \neq \hat{Z})$;
- $(\exists! \hat{X})(\Phi) := (\exists \hat{Y})(\forall \hat{X})(\hat{Y} = \hat{X} \leftrightarrow \Phi)$;
- $(\hat{X}, \ldots, \hat{Y} \in \hat{Z}) := (\hat{X} \in \hat{Z}) \wedge \ldots \wedge (\hat{Y} \in \hat{Z})$;
- $(\hat{X}, \ldots, \hat{Y} \notin \hat{Z}) := (\hat{X} \notin \hat{Z}) \wedge \ldots \wedge (\hat{Y} \notin \hat{Z})$;
- $(\forall \hat{X}, \hat{Y}, \ldots, \hat{Z}) := (\forall \hat{X})(\forall \hat{Y}) \ldots (\forall \hat{Z})$;
- $(\exists \hat{X}, \hat{Y}, \ldots, \hat{Z}) := (\exists \hat{X})(\exists \hat{Y}) \ldots (\exists \hat{Z})$;
- $\hat{Z} = \{\hat{X}, \ldots, \hat{Y}\} := (\forall \hat{W})(\hat{W} \in \hat{Z} \leftrightarrow \hat{W} = \hat{X} \vee \ldots \vee \hat{W} = \hat{Y})$.

Definition 2.1. *A formula of σ-LST is called* ***atomic*** *if it is build according to the following rules:*

(1) $\hat{X} \in \hat{Y}$ *is an atomic formula and means that* $\hat{X}$ *is a σ-element of* $\hat{Y}$*;*
(2) $\neg\Phi$*,* $\Phi \to \Psi$*,* $(\forall \hat{X})(cto\hat{X} \to \Phi)$ *are atomic formulas if* Φ *and* Ψ *are atomic formulas.*

Now, a formula of σ-LST is called ***normal*** *if it is equivalent to an atomic formula. Therefore we obtain that*

$$(\Phi \wedge \Psi),\ (\Phi \vee \Psi),\ (\neg\Phi),\ (\Phi \to \Psi),\ (\Phi \leftrightarrow \Psi),\ (\forall \hat{X})(\Phi),\ (\exists \hat{X})(\Phi),$$

are normal formulas if Φ *and* Ψ *are normal formulas. (In general we use capital Greek letters to denote formulas of σ-LST). The notions of free and bounded variable are defined as usual. A sentence is a formula with no free variables.*

3. Axioms and Theorems

In this section we present the axiom system of σ-Set Theory.

3.1. **The Axiom of Empty σ-set.** There exists a σ-set which has no σ-elements, that is

$$(\exists X)(\forall x)(x \notin X).$$

Definition 3.1. *The σ-set with no σ-elements is called* ***empty σ-set*** *and is denoted by* $\emptyset$*.*

3.2. **The Axiom of Extensionality.** For all σ-classes $\hat{X}$ and $\hat{Y}$, if $\hat{X}$ and $\hat{Y}$ have the same σ-elements, then $\hat{X}$ and $\hat{Y}$ are equal, that is

$$(\forall \hat{X}, \hat{Y})[(\forall z)(z \in \hat{X} \leftrightarrow z \in \hat{Y}) \to \hat{X} = \hat{Y}].$$

Theorem 3.2. *The empty σ-set is unique.*

3.3. **The Axiom of Creation of σ-Class.** We consider an atomic formula $\Phi(x)$ (where $\hat{Y}$ is not free). Then there exists the classes $\hat{Y}$ of all σ-sets that satisfies $\Phi(x)$, that is

$$(\exists \hat{Y})(x \in \hat{Y} \leftrightarrow \Phi(x)),$$

with $\Phi(x)$ a atomic formula where $\hat{Y}$ is not free.

Theorem 3.3. *If $\Phi(x)$ is a normal formula, then there exists σ-class $\hat{Y}$ of all σ-sets that satisfies $\Phi(x)$ and this is unique. That is, if $\Phi(x)$ is a normal formula*

$$(\exists!\hat{Y})(\forall x)(x \in \hat{Y} \leftrightarrow \Phi(x)).$$

Therefore, by Theorem 3.3 if $\Phi(x)$ is a normal formula, we say that

$$\hat{Y}_\Phi = \{x : \Phi(x)\},$$

is the σ-class of all σ-sets that satisfies $\Phi(x)$. Other relevant Theorem respect to de creation of the σ-class it is the following.

Theorem 3.4. *If $\Phi(x)$ is a normal formula, then*

$$(\forall x)(x \in \hat{Y}_\Phi \leftrightarrow \Phi(x)).$$

It is clear that if $\Phi(x)$ is a normal formula, then $\hat{Y}_\Phi$ is a normal term. Hence, the empty σ-class and universal σ-class we can define as

$$\Theta = \{x : x \neq x\}, \quad \mathbf{U} = \{x : x = x\}.$$

Definition 3.5. *Let $\hat{X}$ and $\hat{Y}$ be σ-class. Then*

(1) *We define the intersection of $\hat{X}$ and $\hat{Y}$ as*

$$\hat{X} \cap \hat{Y} = \{x : x \in \hat{X} \wedge x \in \hat{Y}\};$$

(2) *If $\hat{X}$ and $\hat{Y}$ are proper σ-class, we define fusion of $\hat{X}$ and $\hat{Y}$ as*

$$\hat{X} \cup \hat{Y} = \{x : x \in \hat{X} \vee x \in \hat{Y}\}.$$

The difference between the fusion of σ-classes and σ-sets we will be see in Axiom 3.15 (Fusion).

3.4. **The Axiom of Scheme of Replacement.** The image of a σ-set under a normal functional formula Φ is a σ-set.

For each normal formula $\Phi(x, y)$, the following normal formula is an Axiom (of Replacement):

$$(\forall x)(\exists!y)(\Phi(x, y)) \to (\forall X)(\exists Y)(\forall y)(y \in Y \leftrightarrow (\exists x \in X)(\Phi(x, y))).$$

Theorem 3.6. *(Schema of Separation) Let Φ be a normal formula. Then for all σ-set X there exists a unique σ-set Y such that $x \in Y$ if and only if $x \in X$ and $\Phi(x)$, that is*

$$(\forall X)(\exists!Y)(\forall x)(x \in Y \leftrightarrow x \in X \wedge \Phi(x)).$$

Definition 3.7. *Let X and Y be σ-sets. We define the following operations on σ-sets.*

(1) $X \cap Y = \{x \in X : x \in Y\}$.
(2) $X - Y = \{x \in X : x \notin Y\}$.

By Theorem 3.6 (Schema of Separation) it is clear that $X \cap Y$ and $X - Y$ are σ-sets.

3.5. **The Axiom of Pairs.** For all X and Y σ-sets there exists a σ-set Z, called fusion of pairs of X and Y, that satisfy one and only one of the following conditions:

- **(a):** Z contains exactly X and Y,
- **(b):** Z is equal to the empty σ-set,

that is

$$(\forall X,Y)(\exists Z)(\forall a)[(a\in Z \leftrightarrow a=X \vee a=Y)\underline{\vee}(a\notin Z)].$$

Notation 3.8. *Let X and Y be σ-sets. The fusion of pairs of X and Y will be denoted by $\{X\}\cup\{Y\}$.*

Lemma 3.9. *If X and Y are σ-sets, then the fusion of pairs of X and Y is unique.*

Proof. Assume that $\{X\}\cup\{Y\}$ satisfies the condition (a) of Axiom 3.5 (Fusion of Pairs) then by Axiom 3.2 (Extensionality) $\{X\}\cup\{Y\}$ is unique. Otherwise, if $\{X\}\cup\{Y\}=\emptyset$, then by Theorem 3.2, we will have that it is unique. $\square$

Theorem 3.10. *If X and Y are σ-sets and the fusion of pairs of X and Y satisfies condition (a) of Axiom 3.5, then*

$$\{X\}\cup\{Y\}=\{X,Y\}.$$

The proof of Theorems 3.2, 3.3, 3.4, 3.6 and 3.10 are standard in Set Theory so we will not include them.

Also it is important to observe that, given a σ-class $\hat{X}$, if there exist $x,y,z,\ldots,u,w$ σ-sets such that

$$(x\in y)\wedge(y\in z)\wedge\ldots\wedge(u\in w)\wedge(w\in\hat{X}),$$

then it will be possible to introduce the concept of ϵ-chain of $\hat{X}$. ϵ-Chains will be useful in two different ways:

1. To distinguish when a σ-set cannot be constructed from the empty σ-set.
2. To distinguish when two σ-sets are totally different.

We use the following abbreviations

- $\triangleleft x,y,z\triangleright := x\in y\in z := (x\in y\wedge y\in z)$;
- $u\in\triangleleft x,y,z\triangleright := (u=x\vee u=y\vee u=z)$;
- $u\notin\triangleleft x,y,z\triangleright := (u\neq x\wedge u\neq y\wedge u\neq z)$;
- $\triangleleft x,y,w\triangleright \neq \triangleleft a,b,c\triangleright := (\exists u,v)[u\in\triangleleft x,y,w\triangleright\wedge v\in\triangleleft a,b,c\triangleright\wedge u\neq v]$;
- $\triangleleft x,y,w\triangleright \not\equiv \triangleleft a,b,c\triangleright := (\forall u,v)[u\in\triangleleft x,y,w\triangleright\wedge v\in\triangleleft a,b,c\triangleright\to u\neq v]$;
- $\triangleleft x,y,z\triangleright\in CH(\hat{X}) := (x\in y\in z\in\hat{X})$;
- $\exists\triangleleft x,y,z\triangleright\in CH(\hat{X}) := (\exists x,y,z)(x\in y\in z\in\hat{X})$;
- $(\exists\triangleleft x,y,z\triangleright\in CH(\hat{X}))(\Phi) := (\exists x,y,z)(x\in y\in z\in\hat{X}\wedge\Phi)$;
- $\forall\triangleleft x,y,z\triangleright\in CH(\hat{X}) := (\forall x,y,z)(x\in y\in z\in\hat{X})$;
- $(\forall\triangleleft x,y,z\triangleright\in CH(\hat{X}))(\Phi) := (\forall x,y,z)(x\in y\in z\in\hat{X}\to\Phi)$;

- $\lhd x, y, z \rhd \in CH_p(\hat{X}) := (x \in y \in z \in \hat{X} \wedge x, y \in \hat{X})$;
- $\exists \lhd x, y, z \rhd \in CH_p(\hat{X}) := (\exists x, y, z)(x \in y \in z \in \hat{X} \wedge x, y \in \hat{X})$;
- $(\exists \lhd x, y, z \rhd \in CH_p(\hat{X}))(\Phi) := (\exists x, y, z)(x \in y \in z \in \hat{X} \wedge x, y \in \hat{X} \wedge \Phi)$;
- $\forall \lhd x, y, z \rhd \in CH_p(\hat{X}) := (\forall x, y, z)(x \in y \in z \in \hat{X} \wedge x, y \in \hat{X})$;
- $(\forall \lhd x, y \rhd \in CH_p(\hat{X}))(\Phi) := (\forall x, y)(x \in y \in \hat{X} \wedge x \in \hat{X} \to \Phi)$.

Also we use the abbreviations

$$(\forall \lhd x, y, z \rhd \in CH(\hat{X}))(\forall \lhd a, b, c, d \rhd \in CH(\hat{Y}))(\Phi) :=$$
$$(\forall x, y, z, a, b, c, d)(x \in y \in z \in \hat{X} \wedge a \in b \in c \in d \in \hat{Y} \to \Phi).$$

Note that the above formulas can be extended to more variables.

Regarding ϵ-chains, the next definitions will be relevant.

Definition 3.11. *Let $\hat{X}$ be a σ-class and $x, \ldots, w$ be σ-sets. We say that:*

(a): $\lhd x, \ldots, w \rhd$ *is an* ***ϵ-chain of*** $\hat{X}$ *if* $\lhd x, \ldots, w \rhd \in CH(\hat{X})$*;*

(b): $\lhd x, \ldots, w \rhd$ *is a* ***proper ϵ-chain of*** $\hat{X}$ *if* $\lhd x, \ldots, w \rhd \in CH_p(\hat{X})$.

Definition 3.12. *Let $\hat{X}$ be a σ-class and $x, \ldots, w$ be σ-sets such that*

$$\lhd x, \ldots, w \rhd \in CH(\hat{X}).$$

We say that u is a ***link of the ϵ-chain*** $\lhd x, \ldots, w \rhd$ *if* $u \in \lhd x, \ldots, w \rhd$.

In particular, if there exist $x, \ldots, w$ be σ-sets such that $\lhd x, \ldots, w \rhd \in CH(\hat{X})$ we will say that the σ-set x is the **least link** and the σ-set w is the **greatest link** of the ϵ-chain.

Let us introduce the abbreviation

$$u, \ldots, v \in \lhd x, \ldots, w \rhd := u \in \lhd x, \ldots, w \rhd \wedge \ldots \wedge v \in \lhd x, \ldots, w \rhd .$$

Definition 3.13. *Let $\hat{X}$ be a σ-class. If there exist $\lhd x, \ldots, w \rhd, \lhd a, \ldots, c \rhd \in CH(\hat{X})$ we say that:*

(1) $\lhd x, \ldots, w \rhd$ ***is different from*** $\lhd a, \ldots, c \rhd$ *if and only if*
$$\lhd x, \ldots, w \rhd \neq \lhd a, \ldots, c \rhd;$$
(2) $\lhd x, \ldots, w \rhd$ ***is totally different from*** $\lhd a, \ldots, c \rhd$ *if and only if*
$$\lhd x, \ldots, w \rhd \not\equiv \lhd a, \ldots, c \rhd;$$
(3) $\lhd x, \ldots, w \rhd$ ***is an extending ϵ-chain of*** $\hat{X}$ *if the least link is non-empty (i.e. $x \neq \emptyset$).*

Also we will say that two ϵ-chains are disjoint if they are totally different.

Definition 3.14. *Let X and Y be nonempty σ-sets. We will say that X and Y* ***are totally different*** *($X \not\equiv Y$) if any ϵ-chain of X is disjoint to any ϵ-chain of Y. That is*

$$(\forall \lhd x, \ldots, w \rhd \in CH(X))(\forall \lhd a, \ldots, c \rhd \in CH(Y))(\lhd x, \ldots, w \rhd \not\equiv \lhd a, \ldots, c \rhd).$$

Theorem 3.15. *Let X and Y be nonempty σ-sets. If $X \not\equiv Y$, then*

(a): $X \cap Y = \emptyset$.

(b): $X \notin Y \wedge Y \notin X$.

Proof. **(a):** Suppose that $X \cap Y \neq \emptyset$ then there exists $a \in X \cap Y$. Therefore $\triangleleft a \triangleright \in CH_p(X)$ and $\triangleleft a \triangleright \in CH_p(Y)$, which is a contradiction because $X \not\equiv Y$.

(b): Now suppose that $X \in Y$. Since X and Y are nonempty then there exists a σ-element $a \in X$ and so $\triangleleft a \triangleright \in CH(X)$. It the same way, $\triangleleft a, X \triangleright \in CH(Y)$, which is a contradiction since $X \not\equiv Y$. The proof that $Y \notin X$ is analogous.

□

Definition 3.16. *Let X be a nonempty σ-set. Then we say that:*

(1) X ***is non constructible from the empty σ-set*** *if*

$$(\forall \triangleleft x, \ldots, w \triangleright \in CH(X))(x \neq \emptyset).$$

That is $\triangleleft x, \ldots, w \triangleright$ is an extending ϵ-chain of X.

(2) X ***has the linear ϵ-root property*** *if*

$$(\forall \triangleleft x, \ldots, w \triangleright \in CH(X))(\forall u)(u \in \triangleleft x, \ldots, w \triangleright \to (\exists! y)(y \in u)).$$

In order to denote the σ-sets that are not constructible from empty σ-set we will use the following σ-class:

$$NC(\emptyset) = \{X : X \text{ is non constructible from the } \emptyset\},$$

and in order to denote the σ-sets that have the linear ϵ-root property we will use the following σ-class:

$$LR = \{X : X \text{ has the linear } \epsilon\text{-root property}\}.$$

Theorem 3.17. *Let X be a nonempty σ-set. If $X \in LR$, then $X \in NC(\emptyset)$.*

Proof. Consider $X \in LR$ and $\triangleleft x, \ldots, w \triangleright \in CH(X)$. Since $X \in LR$ then there exists a unique σ-set y such that $y \in x$. Therefore $x \neq \emptyset$. Finally $X \in NC(\emptyset)$. □

Theorem 3.18. *Let X be a σ-set. If $X \in NC(\emptyset)$, then for all $\triangleleft x, \ldots, w \triangleright \in CH(X)$, there exists a nonempty σ-set y such that $\triangleleft y, x, \ldots, w \triangleright \in CH(X)$.*

Proof. Let $\triangleleft x, \ldots, w \triangleright \in CH(X)$. Since X is non constructible from the empty σ-set then $x \neq \emptyset$. Therefore there exists $y \in x$ and so $\triangleleft y, x, \ldots, w \triangleright \in CH(X)$. Finally we obtain that $y \neq \emptyset$, because X is non constructively from the empty σ-set. □

Corollary 3.19. *If $X \in NC(\emptyset)$, then $\emptyset \notin X$.*

Proof. This fact is obvious by Theorem 3.18. □

3.6. **The Axiom of Weak Regularity.** For all σ-set X, for all $\lhd x, \ldots, w \rhd \in CH(X)$ we have that $X \notin \lhd x, \ldots, w \rhd$, that is

$$(\forall X)(\forall \lhd x, \ldots, w \rhd \in CH(X))(X \notin \lhd x, \ldots, w \rhd).$$

Theorem 3.20. *Let X be a σ-set. Then X is not a σ-element of X.*

Proof. Let X be a σ-set. It is clear that if $X = \emptyset$ then $X \notin X$. Now, suppose that $X \neq \emptyset$ and $X \in X$. Then $\lhd X \rhd \in CH(X)$, which is a contradiction with Axiom 3.6 (Weak Regularity). $\square$

Theorem 3.21. *Let X and Y be σ-sets. If X is a σ-element of Y then Y is not a σ-element of X.*

Proof. Consider X and Y σ-sets such that $X \in Y$ and $Y \in X$. Then we can obtain $\lhd X, Y \rhd \in CH(X)$, which is a contradiction with Axiom 3.6 (Weak Regularity). $\square$

Definition 3.22. *Let X be a σ-set. We say that X is a **singleton** if there exists a unique σ-set x such that $x \in X$.*

In order to denote the singleton we will use the following σ-class:

$$SG = \{X : X \text{ is a singleton}\}.$$

Now, we will introduce the notion of σ-subclass of $\hat{X}$. We say that $\hat{Y}$ is a σ-subclass of a σ-class $\hat{X}$ if for all $x \in \hat{Y}$ then $x \in \hat{X}$ (i.e. $\hat{Y} \subseteq \hat{X}$). It is clear that $\hat{X} = \hat{Y}$ if and only if $\hat{X} \subseteq \hat{Y}$ and $\hat{Y} \subseteq \hat{X}$.

Definition 3.23. *Let $\hat{X}$ be a σ-class. We define:*

- **(a):** *Given $y \in \hat{X}$, we say that y **is an ϵ-minimal σ-element of** $\hat{X}$ if for all $\lhd x, \ldots, w \rhd \in CH(y)$ we have that $x, \ldots, w \notin \hat{X}$.*
- **(b):** *The **ϵ-minimal σ-subclass of** $\hat{X}$ is*

$$\min(\hat{X}) = \{y \in \hat{X} : y \textit{ is an } \epsilon\textit{-minimal } \sigma\textit{-element of } \hat{X}\}.$$

- **(c):** *Given $y \in \hat{X}$, we say that y **is an ϵ-maximal σ-element of** $\hat{X}$ if for all $z \in \hat{X}$ and for all $\lhd x, \ldots, w \rhd \in CH(z)$ we have that $y \notin \lhd x, \ldots, w \rhd$.*
- **(d):** *The **ϵ-maximal σ-subclass of** $\hat{X}$ is*

$$\max(\hat{X}) = \{y \in \hat{X} : y \textit{ is an } \epsilon\textit{-maximal } \sigma\textit{-element of } \hat{X}\}.$$

Therefore we obtain that:

$$y \in \min(\hat{X}) \leftrightarrow (y \in \hat{X}) \wedge (\forall \lhd x, \ldots, w \rhd \in CH(y))(x, \ldots, w \notin \hat{X});$$
$$y \notin \min(\hat{X}) \leftrightarrow (y \notin \hat{X}) \vee (\exists \lhd x, \ldots, w \rhd \in CH(y))(x \in \hat{X} \vee \ldots \vee w \in \hat{X});$$
$$y \in \max(\hat{X}) \leftrightarrow (y \in \hat{X}) \wedge (\forall z \in \hat{X})(\forall \lhd x, \ldots, w \rhd \in CH(z))(y \notin \lhd x, \ldots, w \rhd);$$
$$y \notin \max(\hat{X}) \leftrightarrow (y \notin \hat{X}) \vee (\exists z \in \hat{X})(\exists \lhd x, \ldots, w \rhd \in CH(z))(y \in \lhd x, \ldots, w \rhd).$$

It is clear that if X is a σ-set then $\min(X)$ and $\max(X)$ are σ-subsets of X.

3.7. **The Axiom of non ϵ-Bounded σ-Set.** There exists a non ϵ-bounded σ-set, that is

$$(\exists X)(\exists y)[(y \in X) \wedge (\min(X) = \emptyset \vee \max(X) = \emptyset)].$$

Definition 3.24. *Let X be a nonempty σ-set. We say that:*

(1) *X is lower ϵ-bounded if and only if* $\min(X) \neq \emptyset$*;*
(2) *X is upper ϵ-bounded if and only if* $\max(X) \neq \emptyset$*;*
(3) *X is ϵ-bounded if and only if X is a lower ϵ-bounded and upper ϵ-bounded.*

Also, we say that a σ-set X is ϵ-finite if is ϵ-bounded and ϵ-infinite if is non ϵ-bounded. Therefore, in order to denote the σ-class of all ϵ-infinite σ-sets we will use the following notation:

$$IF = \{X : \min(X) = \emptyset \vee \max(X) = \emptyset\},$$

and in order to denote the σ-class of all ϵ-finite σ-sets we will use

$$FN = \{X : \min(X) \neq \emptyset \wedge \max(X) \neq \emptyset\}.$$

Lemma 3.25. *Let X be a σ-set. If $X \in SG$, then the following statement holds:*

(a): *X is ϵ-finite (i.e. $X \in FN$).*
(b): $\min(X) = \max(X) = X$.

Proof. We consider $X = \{x\}$.

(a): Suppose that $\min(X) = \emptyset$. Therefore $x \notin \min(X)$. So, there exists $\lhd y, \ldots, w \rhd \in CH(x)$ such that $y \in X \vee \ldots \vee w \in X$. In consequence there exists $u \in \lhd y, \ldots, w \rhd$ such that $u \in X$. Since $X = \{x\}$ then $u = x$. Finally, we have that there exists $\lhd y, \ldots, w \rhd \in CH(x)$ such that $x \in \lhd y, \ldots, w \rhd$ which is a contradiction with Axiom 3.6 (Weak Regularity). So, $min(X) \neq \emptyset$.

Now, if $\max(X) = \emptyset$ then $x \notin \max(X)$. Therefore, there exist $z \in X$ and $\lhd y, \ldots, w \rhd \in CH(z)$ such that $x \in \lhd y, \ldots, w \rhd$. Since $X = \{x\}$ then $z = x$. So there exists $\lhd y, \ldots, w \rhd \in CH(x)$ such that $x \in \lhd y, \ldots, w \rhd$ which is a contradiction with Axiom 3.6 (Weak Regularity). So, $max(X) \neq \emptyset$.

(b): (b.1) We prove that $\min(X) = X$. It is clear, from Definition 3.23 that $\min(X) \subseteq X$. Since $\min(X) \neq \emptyset$ then there exists $y \in \min(X)$ and so $y \in X$. Therefore $y = x$ because $X = \{x\}$. So $\min(X) = X$.

(b.2) We prove that $\max(X) = X$. It is clear, from Definition 3.23 that $\max(X) \subseteq X$. Since $\max(X) \neq \emptyset$ then there exists $y \in \max(X)$ and so $y \in X$. Therefore $y = x$ because $X = \{x\}$. So $\max(X) = X$.

Finally by Axiom 3.2 (Extensionality) we have that $\min(X) = \max(X) = X$.

$\square$

3.8. **The Axiom of Weak Choice.** If $\hat{X}$ be a σ-class of σ-sets, then we can choose a singleton Y whose unique σ-element come from $\hat{X}$, that is

$$(\forall \hat{X})(\forall x)(x \in \hat{X} \to (\exists Y)(Y = \{x\})).$$

This axiom guarantees the existence of 1 and 1^* σ-sets, which will serve to build the σ-antielements and σ-antisets.

3.9. **The Axiom of ϵ-Linear σ-set.** There exists a nonempty σ-set X which has the linear ϵ-root property, that is

$$(\exists X)(\exists y)(y \in X \wedge X \in LR).$$

Definition 3.26. *Let X be a nonempty σ-set. Then X is called a* ***ϵ-linear σ-set****, if X has the linear ϵ-root property (i.e. $X \in LR$).*

It is clear by Axiom 3.9 that there exists $X \in LR$. So if we consider the σ-class X_{LK} of all link of ϵ-chains of X do not have an ϵ-minimal element.

$$X_{LK} = \{u : u \in \triangleleft x, \ldots, w \triangleright \text{ for some } \triangleleft x, \ldots, w \triangleright \in CH(X)\}$$

In fact, suppose that there exists $y \in \min(X_{LK})$. Then $y \in X_{LK}$ and for all $\triangleleft x, \ldots, w \triangleright \in CH(y)$ we have that $x, \ldots, w \notin X_{LK}$, in particular for all $z \in y$ we have that $z \notin X_{LK}$ which is a contradiction. So $\min(X_{LK}) = \emptyset$.

Now, we can consider $X_{LK} \in IF$ and so X_{LK} is a σ-set. The existence of σ-set X_{LK} is guaranteed by Axiom 3.7 (non ϵ-Bounded σ-Set). Also, we observe that $X \notin X_{LK}$ by Axiom 3.6 (Weak Regularity).

Also, it is important to observe that if $X \in LR$ then we can not declare that $X \in SG$, because if $\triangleleft x, \ldots, w \triangleright \in CH(X)$ then $X \notin \triangleleft x, \ldots, w \triangleright$ by Axiom 3.6 (Weak Regularity). In the example 3.30 we will see that the σ-set 1_Γ has the linear ϵ-root property but it is not a singleton. Also, in the same example we will see that the σ-set $X = \{2\}$ is a singleton but X does not have the linear ϵ-root property. However, if $X \in LR$ and $y \in X_{LK}$ then $y \in SG \cap LR$.

Lemma 3.27. *Let X be a nonempty σ-set such that $X \in LR$. If $y \in X_{LK}$, then $y \in SG \cap LR$.*

Proof. Let X a nonempty σ-set such that $X \in LR$. Since $X \neq \emptyset$ then $X_{LK} \neq \emptyset$. Therefore we consider $y \in X_{LK}$. By Definition 3.16 we have that $y \in SG$. Now, we consider $\triangleleft x, \ldots, w \triangleright \in CH(y)$ and $u \in \triangleleft x, \ldots, w \triangleright$. Since $y \in X_{LK}$ then $x, \ldots, w \in X_{LK}$. Therefore $u \in X_{LK}$. Since $X \in LR$ by Definition 3.16 $u \in SG$. In consequence $y \in LR$. Finally $y \in SG \cap LR$. $\square$

Let X be a σ-set, X is called ϵ-linear singleton if $X \in SG \cap LR$.

3.10. **The Axiom of One and One* σ-set.** For all ϵ-linear singleton, there exists a ϵ-linear singleton Y such that X is totally different from Y, that is

$$(\forall X \in SG \cap LR)(\exists Y \in SG \cap LR)(X \not\equiv Y).$$

Let X be a σ-set, in order to denote the σ-class of all σ-sets totally different to X we will use the following notation:

$$TD(X) = \{Y : Y \not\equiv X\}.$$

Now, we introduce the concept to One and One* σ-sets. We consider the following σ-class

$$SG \cap LR = \{X : X \text{ is a singleton and has the linear } \epsilon\text{-root property }\}.$$

By Axiom 3.9 (ϵ-Linear σ-Set) it is clear that $SG \cap LR \neq \Theta$. Therefore, by Axiom 3.8 (Weak Choice) we can choose a σ-set 1 whose unique σ-element come from the σ-class $SG \cap LR$. Therefore, we can define the One σ-set. The unique σ-element of 1 is denoted for α. Therefore

$$1 = \{\alpha\},$$

where $1, \alpha \in SG \cap LR$.

Following the same argument, we consider the σ-class

$$TD(\alpha) = \{Y : Y \not\equiv \alpha\}.$$

Since $\alpha \in SG \cap LR$ by Axiom 3.10 (One and One*) we obtain that $TD(\alpha) \neq \Theta$. Therefore by Axiom 3.8 (Weak Choice) we choose a σ-set 1^* whose unique σ-element come from the σ-class $TD(\alpha)$. Therefore, we can define the One* σ-set. The unique σ-element of 1^* is denoted for β. So

$$1^* = \{\beta\},$$

where $1^*, \beta \in SG \cap LR$. Therefore we obtain that $1, 1^*, \alpha, \beta \in SG \cap LR$.

Now, we present the basic properties that have the σ-sets 1 and 1^*,

- 1 and 1^* are unique;
- $1 \not\equiv 1^* \wedge 1 \not\equiv \beta \wedge \alpha \not\equiv \beta \wedge \alpha \not\equiv 1^*$;
- $\min(\alpha) = \alpha \wedge \min(\beta) = \beta$;
- $\min(1) = 1 \wedge \min(1^*) = 1^*$.

Since fusion of pairs can satisfy the conditions (a) and (b) of Axiom 3.5 then we must precise when the fusion of pairs satisfies one or the other condition. To this end we give the Axioms of Completeness (A) and (B).

Let us introduce the abbreviations

- $\min(X, Y) = |A \wedge B| := (\min(X) = A \wedge \min(Y) = B)$;
- $\min(X, Y) \neq |A \wedge B| := (\min(X) \neq A \wedge \min(Y) \neq B)$;
- $\min(X, Y) = |A \vee B| := (\min(X) = A \vee \min(Y) = B)$;
- $\min(X, Y) \neq |A \vee B| := (\min(X) \neq A \vee \min(Y) \neq B)$.

We define the formula

$$\Psi(z,w,a,x) := (\exists! w)(\{z\}\cup\{w\}=\emptyset)\wedge(\forall a)(\{z\}\cup\{a\}=\emptyset\to a\in x).$$

3.11. **The Axiom of Completeness (A)..** If X and Y are σ-sets, then

$$\{X\}\cup\{Y\}=\{X,Y\},$$

if and only if X and Y satisfy one of the following conditions:

(a): $\min(X,Y)\neq|1\vee 1^*|\wedge\min(X,Y)\neq|1^*\vee 1|$.
(b): $\neg(X\not\equiv Y)$.
(c): $(\exists w\in X)[w\notin\min(X)\wedge\neg\Psi(z,w,a,Y)]$.
(d): $(\exists w\in Y)[w\notin\min(Y)\wedge\neg\Psi(z,w,a,X)]$.

Lemma 3.28. *Let X be a σ-set. Then $\{X\}\cup\{X\}=\{X\}$.*

Proof.

(a): If $X=\emptyset$, then $\min(X)=\emptyset$. Therefore $\min(\emptyset)\neq 1$ and $\min(\emptyset)\neq 1^*$. So, $X=\emptyset$ satisfies the condition (a) of Axiom 3.11 (Completeness A). In consequence $\{\emptyset\}\cup\{\emptyset\}=\{\emptyset\}$.
(b): Now, if $X\neq\emptyset$ then it is clear that X is not totally different from X, then X satisfies the condition (b) of Axiom 3.11 (Completeness A). So, $\{X\}\cup\{X\}=\{X\}$.

□

From now on, we will denote by 1_Θ the σ-set whose only element is $\emptyset$.

Theorem 3.29. *If X is a σ-set, then*

(a): $\{\emptyset\}\cup\{X\}=\{\emptyset,X\}$.
(b): $\{\alpha\}\cup\{X\}=\{\alpha,X\}$.
(c): $\{\beta\}\cup\{X\}=\{\beta,X\}$.

Proof. It is clear that

$$\min(\alpha)\neq 1\wedge\min(\alpha)\neq 1^*,$$
$$\min(\beta)\neq 1\wedge\min(\beta)\neq 1^*,$$
$$\min(\emptyset)\neq 1\wedge\min(\emptyset)\neq 1^*.$$

Therefore the proofs of (a), (b) and (c) are obvious. □

Now, let us see some examples where Axiom 3.11 (Completeness A) can be applied.

Example 3.30.

(1) $\{\emptyset\}\cup\{\emptyset\}=\{\emptyset\}=1_\Theta$,
(2) $\{\emptyset\}\cup\{\alpha\}=\{\emptyset,\alpha\}=1_\Lambda$,
(3) $\{\emptyset\}\cup\{\beta\}=\{\emptyset,\beta\}=1_\Omega$,
(4) $\{\alpha\}\cup\{\alpha\}=\{\alpha\}=1$,

(5) $\{\alpha\} \cup \{\beta\} = \{\alpha, \beta\} = 1_\Gamma$,
(6) $\{\beta\} \cup \{\beta\} = \{\beta\} = 1^*$,
(7) $\{\emptyset\} \cup \{1_\Theta\} = \{\emptyset, 1_\Theta\} = 2_\Theta$,
(8) $\{\emptyset\} \cup \{1_\Lambda\} = \{\emptyset, 1_\Lambda\} = 2_{(\emptyset,\Lambda)}$,
(9) $\{\emptyset\} \cup \{1_\Omega\} = \{\emptyset, 1_\Omega\} = 2_{(\emptyset,\Omega)}$,
(10) $\{\emptyset\} \cup \{1_\Gamma\} = \{\emptyset, 1_\Gamma\} = 2_{(\emptyset,\Gamma)}$,
(11) $\{\emptyset\} \cup \{1\} = \{\emptyset, 1\} = 2_\Theta$,
(12) $\{\emptyset\} \cup \{1^*\} = \{\emptyset, 1^*\} = 2^*_\Theta$,
(13) $\{\alpha\} \cup \{1_\Theta\} = \{\alpha, 1_\Theta\} = 2_{(\alpha,\Theta)}$,
(14) $\{\alpha\} \cup \{1_\Lambda\} = \{\alpha, 1_\Lambda\} = 2_{(\alpha,\Lambda)}$,
(15) $\{\alpha\} \cup \{1_\Omega\} = \{\alpha, 1_\Omega\} = 2_{(\alpha,\Omega)}$,
(16) $\{\alpha\} \cup \{1_\Gamma\} = \{\alpha, 1_\Gamma\} = 2_{(\alpha,\Gamma)}$,
(17) $\{\alpha\} \cup \{1\} = \{\alpha, 1\} = 2$,
(18) $\{\alpha\} \cup \{1^*\} = \{\alpha, 1^*\} = 2^*_\alpha$,
(19) $\{\beta\} \cup \{1_\Theta\} = \{\beta, 1_\Theta\} = 2_{(\beta,\Theta)}$,
(20) $\{\beta\} \cup \{1_\Lambda\} = \{\beta, 1_\Lambda\} = 2_{(\beta,\Lambda)}$,
(21) $\{\beta\} \cup \{1_\Omega\} = \{\beta, 1_\Omega\} = 2_{(\beta,\Omega)}$,
(22) $\{\beta\} \cup \{1_\Gamma\} = \{\beta, 1_\Gamma\} = 2_{(\beta,\Gamma)}$,
(23) $\{\beta\} \cup \{1\} = \{\beta, 1\} = 2_\beta$,
(24) $\{\beta\} \cup \{1^*\} = \{\beta, 1^*\} = 2^*$,
(25) $\{1\} \cup \{2\} = \{1, 2\}$,
(26) $\{1^*\} \cup \{2^*\} = \{1^*, 2^*\}$,

Results from (1) to (24) of Example 3.30 follow from Theorem 3.29 and therefore the condition (a) of Axiom 3.11 (Completeness (A)) holds. Results (25) and (26) follow from condition (b) of Axiom 3.11.

Now, we present the following schemas. Let us introduce the following notation

- $x \to y := x \in y$.

$$
\begin{array}{ccccccc}
 & & & & \{2\} & & \\
 & & & & \uparrow & & \\
 & & & & 2 & \to & \{1,2\} \\
 & & & \nearrow & \uparrow & \nearrow & \\
\cdots & \to & \alpha & \to & 1 & \to & \{1, 1_\Gamma\} \\
 & & & \searrow & & \nearrow & \\
 & & & & 1_\Gamma & \to & \{1_\Gamma\} \\
 & & & \nearrow & & \searrow & \\
\cdots & \to & \beta & \to & 1^* & \to & \{1^*, 1_\Gamma\} \\
 & & & \searrow & \downarrow & \searrow & \\
 & & & & 2^* & \to & \{1^*, 2^*\} \\
 & & & & \downarrow & & \\
 & & & & \{2^*\} & &
\end{array}
$$

It is clear that

- $\min(1_\Gamma) = 1_\Gamma$,
- $1, 1^* \in SG \cap LR$,
- $1_\Gamma \notin SG$ and $1_\Gamma \in LR$,
- $\{2\}, \{2^*\}, \{1_\Gamma\} \in SG$ and $\{2\}, \{2^*\}, \{1_\Gamma\} \notin LR$,
- $\{1, 1_\Gamma\}, \{1^*, 1_\Gamma\}, \{1, 2\}, \{1^*, 2^*\} \notin SG$
- $\{1, 1_\Gamma\}, \{1^*, 1_\Gamma\}, \{1, 2\}, \{1^*, 2^*\} \notin LR$.

$$\begin{array}{ccccccccc}
 & & & & 1 & \rightarrow & 2 & \rightarrow & \cdots \\
 & & & \nearrow & & & & & \\
\cdots & \rightarrow & \alpha & & & & 2_\Lambda & \rightarrow & \cdots \\
 & & & \searrow & & \nearrow & & & \\
 & & & & 1_\Lambda & \rightarrow & 2_{(\emptyset,\Lambda)} & \rightarrow & \cdots \\
 & & & & \uparrow & \nearrow & & & \\
 & & & & \emptyset & \rightarrow & 1_\Theta & \rightarrow & 2_\Theta \\
 & & & & \downarrow & \searrow & & & \\
 & & & & 1_\Omega & \rightarrow & 2_{(\emptyset,\Omega)} & \rightarrow & \cdots \\
 & & & \nearrow & & \searrow & & & \\
\cdots & \rightarrow & \beta & & & & 2_\Omega & \rightarrow & \cdots \\
 & & & \searrow & & & & & \\
 & & & & 1^* & \rightarrow & 2^* & \rightarrow & \cdots
\end{array}$$

3.12. **The Axiom of Completeness (B)..** If X and Y are σ-sets, then

$$\{X\} \cup \{Y\} = \emptyset,$$

if and only if X and Y satisfy the following conditions:

(a): $\min(X, Y) = |1 \wedge 1^*| \vee \min(X, Y) = |1^* \wedge 1|$;
(b): $X \not\equiv Y$;
(c): $(\forall z)(z \in X \wedge z \notin \min(X)) \rightarrow \Psi(z, w, a, Y))$;
(d): $(\forall z)(z \in Y \wedge z \notin \min(Y)) \rightarrow \Psi(z, w, a, X))$.

Definition 3.31. *Let X and Y be σ-sets. If $\{X\} \cup \{Y\} = \emptyset$, then Y* ***is called the σ-antielement of*** *X, and will be denoted by X^*.*

Observe that the basic idea for the inclution of the Axioms 3.11,3.12 (Completeness A and B) come from the following Schemas:

$$\text{(a)} \quad \begin{array}{ccccc}
3 & = \{\alpha, & 1, & 2\ \} & \\
 & & \updownarrow & \updownarrow & \\
4^* & = \{\beta, & 1^*, & 2^*, & 3^*\}
\end{array} \quad \rightarrow \quad \{3\} \cup \{4^*\} = \{3, 4^*\} \quad \text{by AX. 3.11}$$

$$\text{(b)}\quad \begin{array}{lllll} 4 & =\{\alpha, & 1, & 2\,, & 3\,\} \\ & & \updownarrow & \updownarrow & \updownarrow \\ 4^* & =\{\beta, & 1^*, & 2^*, & 3^*\} \end{array} \quad \rightarrow \quad \{4\} \cup \{4^*\} = \emptyset \qquad\qquad \text{by AX. 3.12}$$

Theorem 3.32. *Let X and Y be σ-sets. Then $\{X\} \cup \{Y\} = \emptyset$, if and only if $\{Y\} \cup \{X\} = \emptyset$.*

Proof. $(\rightarrow)$ Suppose that $\{X\} \cup \{Y\} = \emptyset$, then X and Y satisfies the conditions (a), (b), (c) and (d) of Axiom 3.12 (Completeness B). In consequence we obtain the following:

- **(a):** Since $\min(X, Y) = |1 \wedge 1^*|$ or $\min(X, Y) = |1^* \wedge 1|$, then it is clear that $\min(Y, X) = |1^* \wedge 1|$ or $\min(Y, X) = |1 \wedge 1^*|$.
- **(b):** Since $X \not\equiv Y$, then $Y \not\equiv X$ by Definition 3.14.
- **(c):** Let $z \in Y$ such that $z \notin \min(Y)$. Since X and Y satisfy the condition (d) of Axiom 3.12, there exists a unique w such that $\{z\} \cup \{w\} = \emptyset$ and given a if $\{z\} \cup \{a\} = \emptyset$ we have that $a \in X$. In consequence the condition (c) is applies for the fusion of pairs of Y and X.
- **(d):** Following the previous proof, this condition is satisfied. Therefore $\{Y\} \cup \{X\} = \emptyset$.

$(\leftarrow)$ The converse follows in a similar way. □

Corollary 3.33. *If X and Y are σ-sets, then the fusion of pairs of X and Y is commutative.*

Proof. This fact is obvious by Theorems 3.10 and 3.32. □

Regarding σ-antielements, the next results will be relevant.

By Theorem 3.29, we obtain that there exist σ-sets without σ-antielement as the case of the empty σ-set. However, we will prove that, if a σ-set X has a σ-antielement then it is unique.

Theorem 3.34. *Let X be a σ-set. If there exists X^* the σ-antielement of X, then X^* is unique.*

Proof. Consider two σ-sets X and X^* such that $\{X\} \cup \{X^*\} = \emptyset$. Now, suppose that there exists $\widehat{X}$, a σ-set such that $\{X\} \cup \{\widehat{X}\} = \emptyset$ and $X^* \neq \widehat{X}$. Without loss of generality, we assume that $\min(X) = 1$ and $\min(X^*) = 1^*$. Since $\min(X) = 1$ and $\{X\} \cup \{\widehat{X}\} = \emptyset$ then $\min(\widehat{X}) = 1^*$. Therefore, it is clear that $\beta \in X^* \cap \widehat{X}$.

Now, as $X^* \neq \widehat{X}$ there exists $c \in \widehat{X}$ such that $c \notin X^*$ or there exists $a \in X^*$ such that $a \notin \widehat{X}$. If $c \in \widehat{X}$ and $c \notin X^*$ then $c \notin \min(\widehat{X})$. Therefore there exists a unique w, such that $\{c\} \cup \{w\} = \emptyset$. Also, we obtain that $w \in X$. Since

$\{c\} \cup \{w\} = \emptyset$ by Theorems 3.29 and 3.32 we obtain that $w \neq \alpha$. Therefore $w \notin \min(X)$.

On the other hand, since $\{X\} \cup \{X^*\} = \emptyset$, $w \in X$ and $w \notin \min(X)$, we obtain that there exists a unique v such that $\{w\} \cup \{v\} = \emptyset$. So $v \in X^*$. Finally, by Theorem 3.32 we obtain that $\{w\} \cup \{v\} = \{v\} \cup \{w\} = \emptyset$. So $c = v \in X^*$, which is a contradiction.

When there exists $a \in X^*$ such that $a \notin \widehat{X}$ we get the same contradiction. In consequence X^* is unique. $\square$

Corollary 3.35. *Let X be a σ-set. If there exists X^*, the σ-antielement of X, then $(X^*)^* = X$.*

Proof. By Theorem 3.34, it is enough to prove that $\{X^*\} \cup \{X\} = \emptyset$. But this fact is obvious by Theorem 3.32. $\square$

Theorem 3.36. *Consider the σ-sets 1 and 1^*. Then $\{1\} \cup \{1^*\} = \emptyset$.*

Proof. It is clear that $\min(1) = 1$, $\min(1^*) = 1^*$ and $1 \not\equiv 1^*$, then the conditions (a) and (b) of Axiom 3.12 (Completeness B) holds. Now, the conditions (c) and (d) of Axiom 3.12 follow from the fact that $\min(1) = 1$ and $\min(1^*) = 1^*$. $\square$

Theorem 3.37. *Consider the σ-sets 2 and 2^*. Then $\{2\} \cup \{2^*\} = \emptyset$.*

Proof.

- **(a):** It is clear that $2 = \{\alpha, 1\}$ and $2^* = \{\beta, 1^*\}$. Then $\min(2) = 1$ and $\min(2^*) = 1^*$. In consequence the condition (a) of Axiom 3.12 (Completeness (B)) holds.
- **(b):** Now, suppose that 2 is not totally different from 2^*, then there exist $\triangleleft x, \ldots, w \triangleright \in CH(2)$ and $\triangleleft a, \ldots, c \triangleright \in CH(2^*)$ that are not disjoint. Therefore there exists z such that $z \in \triangleleft x, \ldots, w \triangleright$ and $z \in \triangleleft a, \ldots, c \triangleright$. We observe that the greater link of ϵ-chains $\triangleleft x, \ldots, w \triangleright$ and $\triangleleft a, \ldots, c \triangleright$ satisfies the following conditions:
  (1) $w = \alpha \vee w = 1$.
  (2) $c = \beta \vee c = 1^*$.

  Now, if $z = w$ then $z \neq c$ because $1 \not\equiv 1^*$. Therefore we can construct $\triangleleft z, \ldots, \beta \triangleright \in CH(1^*)$, which is a contradiction because $1 \not\equiv 1^*$. In the case that $z = c$ we obtain the same contradiction. Finally, if $z \neq w$ and $z \neq c$, then we can construct $\triangleleft z, \ldots, \alpha \triangleright \in CH(1)$ and $\triangleleft z, \ldots, \beta \triangleright \in CH(1^*)$, which is a contradiction because $1 \not\equiv 1^*$. Therefore we obtain that $2 \not\equiv 2^*$.
- **(c):** Let $z \in 2$ and $z \notin \min(2)$. Since $z \notin \min(2)$ then $z = 1$. By Theorem 3.36 we have that there exists a unique 1^* such that $\{1\} \cup \{1^*\} = \emptyset$. Finally, we consider a such that $\{1\} \cup \{a\} = \emptyset$, then $a = 1^*$, therefore we obtain that $a \in 2^*$.
- **(d):** This proof is analogous to the previous one.

$\square$

Now, it is important to observe that by Theorem 3.29 there exist σ-sets which do not have σ-antielement.

Example 3.38. *As a consequence from Theorem 3.36 and the conditions (c) and (d) of Axiom 3.11 (Completeness A) we have the following results:*

$$\text{(a)}\quad \begin{array}{lll} 1 & = \{\alpha\} & \\ & & \\ 2^* & = \{\beta, & 1^*\} \end{array} \quad \rightarrow \quad \{1\} \cup \{2^*\} = \{1, 2^*\}$$

$$\text{(b)}\quad \begin{array}{lll} 2 & = \{\alpha, & 1\} \\ & & \\ 1^* & = \{\beta\} & \end{array} \quad \rightarrow \quad \{1^*\} \cup \{2\} = \{1^*, 2\}$$

$$\text{(c)}\quad \begin{array}{llll} 2 & = \{\alpha, & 1\ \} & \\ & & \updownarrow & \\ 3^* & = \{\beta, & 1^*, & 2^*\}. \end{array} \quad \rightarrow \quad \{2\} \cup \{3^*\} = \{2, 3^*\}$$

3.13. **The Axiom of Exclusion.** For all σ-sets X, Y, Z, if Y and Z are σ-elements of X then the fusion of pairs of Y and Z contains exactly Y and Z, that is

$$(\forall X, Y, Z)(Y, Z \in X \rightarrow \{Y\} \cup \{Z\} = \{Y, Z\}).$$

Theorem 3.39. *(**Exclusion of Inverses**) Let X be a σ-set. If $x \in X$, then $x^* \notin X$.*

Proof. Assume that $X \neq \emptyset$ and suppose that $x, x^* \in X$, then by Axiom 3.13 (Exclusion) we obtain that $\{x\} \cup \{x^*\} \neq \emptyset$, which is a contradiction. □

One of the most important characteristics of the σ-ST is reflected in the Theorem 3.39. On the other hand, it is clear that $1, 1^* \in SG \cap LR$, hence by Theorem 3.39 we see that $SG \cap LR$ is a proper σ-class. Also, we have that if we define the σ-class

$$AT = \{X : \min(X) = 1 \vee \min(X) = 1^*\}$$

then for all σ-set X such that there exists X^* the σ-antielement of X, we obtain that $X, X^* \in AT$, by condition (a) of Axiom 3.12 (Completeness B). Therefore, by Theorem 3.39 we see that AT is a proper σ-class.

Definition 3.40. *Let X and Y be σ-sets. We define two new operations on σ-sets:*

(1) $X \widehat{\cap} Y := \{x \in X : x^* \in Y\}$;
(2) $X \circledast Y := X - (X \widehat{\cap} Y)$.

By Theorem 3.6 (Schema of Separation) it is clear that $X\widehat{\cap}Y$ and $X \divideontimes Y$ are σ-sets. Also we observe that

$$X\widehat{\cap}Y \subseteq X \wedge Y\widehat{\cap}X \subseteq Y$$

and

$$X \divideontimes Y \subseteq X \wedge Y \divideontimes X \subseteq Y.$$

Example 3.41. *By Example 3.30, we have the following:*

(a)
$$\begin{array}{lll} 2_\Theta & = \{\emptyset, & 1_\Theta\} \\ & & \\ 2 & = \{\alpha, & 1\} \end{array} \quad \rightarrow \quad 2_\Theta\widehat{\cap}2 = \{x \in 2_\Theta : x^* \in 2\} = \emptyset.$$

Therefore $2_\Theta \divideontimes 2 = 2_\Theta$.

(b)
$$\begin{array}{lll} 2_\beta & = \{\beta, & 1\ \} \\ & & \updownarrow \\ 2^* & = \{\beta, & 1^*\} \end{array} \quad \rightarrow \quad 2_\beta\widehat{\cap}2^* = \{x \in 2_\beta : x^* \in 2^*\} = \{1\}.$$

Therefore $2_\beta \divideontimes 2^* = \{\beta\}$.

(c)
$$\begin{array}{lll} A & = \{1, & 2\ \} \\ & \ \ \updownarrow & \updownarrow \\ A^\star & = \{1^*, & 2^*\} \end{array} \quad \rightarrow \quad \begin{array}{c} A\widehat{\cap}A^\star = \{x \in A : x^* \in A^\star\} = A \\ \textit{and} \\ A^\star\widehat{\cap}A = \{x \in A^\star : x^* \in A\} = A^\star. \end{array}$$

Therefore $A \divideontimes A^\star = \emptyset$ *and* $A^\star \divideontimes A = \emptyset$.

Theorem 3.42. *Let X be a σ-set X. Then*

(a): $X\widehat{\cap}\emptyset = \emptyset\widehat{\cap}X = X\widehat{\cap}\alpha = \alpha\widehat{\cap}X = X\widehat{\cap}\beta = \beta\widehat{\cap}X = \emptyset$.
(b): $X\widehat{\cap}1_\Theta = 1_\Theta\widehat{\cap}X = X\widehat{\cap}1 = 1\widehat{\cap}X = X\widehat{\cap}1^* = 1^*\widehat{\cap}X = \emptyset$.
(c): $X\widehat{\cap}X = \emptyset$.

Proof. **(a):** It is clear that $X\widehat{\cap}\emptyset = \emptyset\widehat{\cap}X = \emptyset$ by Definition 3.40. Now, suppose that $X\widehat{\cap}\alpha \neq \emptyset$ or $\alpha\widehat{\cap}X \neq \emptyset$. If $X\widehat{\cap}\alpha \neq \emptyset$, we have that there exist $x \in X$ and $x^* \in \alpha$. By condition (a) of Axiom 3.12 (Completeness B), we obtain that $\min(x^*) = 1$ or $\min(x^*) = 1^*$. If $\min(x^*) = 1$ then $\alpha \in x^*$. Therefore $\alpha \in x^* \in \alpha$ which is a contradiction by Axiom 3.6 (Weak Regularity). Now, if $\min(x^*) = 1^*$ then $\beta \in x^*$, and therefore $\beta \in x^* \in \alpha$, which is a contradiction because $1 \not\equiv 1^*$. Following a similar reasoning, in the case that $\alpha\widehat{\cap}X \neq \emptyset$ we obtain the same contradiction.

The proof that $X\widehat{\cap}\beta = \beta\widehat{\cap}X = \emptyset$ is analogous to the previous one.

(b): This fact is obvious by Theorem 3.29.
(c): This fact is obvious by Theorem 3.39.

□

Corollary 3.43. *Let X be a σ-set X. Then*

(a): $X \divideontimes \emptyset = X$ *and* $\emptyset \divideontimes X = \emptyset$;
(b): $X \divideontimes \alpha = X$ *and* $\alpha \divideontimes X = \alpha$;
(c): $X \divideontimes \beta = X$ *and* $\beta \divideontimes X = \beta$;
(d): $X \divideontimes 1_\Theta = X$ *and* $1_\Theta \divideontimes X = 1_\Theta$;
(e): $X \divideontimes 1 = X$ *and* $1 \divideontimes X = 1$;
(f): $X \divideontimes 1^* = X$ *and* $1^* \divideontimes X = 1^*$;
(g): $X \divideontimes X = X$.

Proof. This proof is obvious by Definition 3.40 and Theorem 3.42. □

Definition 3.44. *Let F be a σ-set. We say that F is **σ-antielement free** if for all $X, Y \in F$ then $X \widehat{\cap} Y = \emptyset$.*

In order to denote a σ-antielement free σ-set we will use the following σ-class:

$$AF = \{X : X \text{ is } \sigma\text{-antielement free }\}.$$

It is clear, by Theorem 3.42, that the σ-set $1_\Gamma = \{\alpha, \beta\} \in AF$. Nevertheless the σ-set $F = \{2_\beta, 2^*\} \notin AF$.

Lemma 3.45. *Let X be a σ-set. Then $\{X\} \in AF$.*

Proof. This fact is obvious by Theorem 3.42. □

3.14. **The Axiom of Power σ-set.** For all σ-set X there exists a σ-set Y, called power of X, whose σ-elements are exactly the σ-subsets of X, that is

$$(\forall X)(\exists Y)(\forall z)(z \in Y \leftrightarrow z \subseteq X).$$

Definition 3.46. *Let X be a σ-set,*

1. *If $Z \subset X$, then Z is a proper σ-subset of X.*
2. *The σ-set of all σ-subsets of X,*
$$2^X = \{z : z \subseteq X\},$$
is called the power σ-set of X.

Theorem 3.47. *Let F be a σ-set. If $F \in AF$ and $X \in 2^F$, then $X \in AF$.*

Proof. We consider $F \in AF$ and $X \in 2^F$. If $X = \emptyset$ then it is clear that $X \in AF$. Now, suppose that $X \neq \emptyset$ and $A, B \in X$. Since $X \in 2^F$ then $A, B \in F$ and therefore $A \widehat{\cap} B = \emptyset$. So $X \in AF$. □

Theorem 3.48. *Let X be a σ-set. Then $2^X \in AF$.*

Proof. Suppose that there exist $A, B \in 2^X$ such that $A \widehat{\cap} B \neq \emptyset$. In consequence there exist $x \in A$ and $x^* \in B$. Therefore $x, x^* \in X$, which is a contradiction by Theorem 3.39. So $2^X \in AF$. □

Corollary 3.49. *Let X be a σ-set. Then $F = \{\{x\} : x \in X\} \in AF$.*

Proof. This proof is obvious by Theorems 3.47 and 3.48. □

3.15. **The Axiom of Fusion.** For all σ-sets X and Y, there exists a σ-set Z, called fusion of all σ-elements of X and Y, such that Z contains σ-elements of the σ-elements of X or Y, that is

$$(\forall X, Y)(\exists Z)(\forall b)(b \in Z \to (\exists z)[(z \in X \vee z \in Y) \wedge (b \in z)]).$$

Now, we can define the fusion of σ-sets.

Definition 3.50. *Let X and Y be σ-sets. Then we define the fusion of X and Y as*

$$X \cup Y = \{x : (x \in X \divideontimes Y) \vee (x \in Y \divideontimes X)\}.$$

It is clear, by Definition 3.50 and Axiom 3.2 (Extensionality), that for all σ-sets X and Y, the fusion of σ-sets is commutative; that is $X \cup Y = Y \cup X$.

Also, we observe that if X^* is the σ-antielement of X then $X, X^* \in AT$ where

$$AT = \{X : \min(X) = 1 \vee \min(X) = 1^*\}.$$

Therefore, by Axiom 3.8 (Weak Choice) we can choose the singleton $F = \{X^*\}$ and $E = \{X\}$ and so by Axiom 3.15 (Fusion) there exists $X \cup X^*$, the fusion of X and X^*. Remember that AT is a proper σ-class by Axiom 3.13 (Exclusion).

Now, it is important to note the difference between the fusion of σ-sets and proper σ-class. Therefore

- If $\hat{X}, \hat{Y}$ are proper σ-classes, then
$$\hat{X} \cup \hat{Y} = \{x : x \in \hat{X} \vee x \in \hat{Y}\}.$$
- If X, Y are σ-sets, then
$$X \cup Y = \{x : x \in X \divideontimes Y \vee x \in Y \divideontimes X\}.$$

In general, we are only interested in the properties of the fusion of σ-sets.

Example 3.51. *It is clear that $1, 1^*, 2, 2^* \in AT$ where AT is a proper σ-class. Therefore by Axiom 3.8 (Weak Choice) we can choose the singleton $A = \{1\}$, $B = \{1^*\}$, $C = \{2\}$ and $D = \{2^*\}$. So by Axiom 3.15 (Fusion) there exists*

(a) $$\begin{aligned} 1 \cup 1^* &= \{x : x \in 1 \divideontimes 1^* \vee x \in 1^* \divideontimes 1\} \\ &= \{x : x \in 1 \vee x \in 1^*\} \\ &= \{\alpha, \beta\} = \{\alpha\} \cup \{\beta\} = 1_\Gamma, \end{aligned}$$

(b) $$\begin{aligned} 2 \cup 2^* &= \{x : x \in 2 \divideontimes 2^* \vee x \in 2^* \divideontimes 2\} \\ &= \{x : x \in 1 \vee x \in 1^*\} \\ &= \{\alpha, \beta\} = \{\alpha\} \cup \{\beta\} = 1_\Gamma, \end{aligned}$$

(c) $$\begin{aligned} 1 \cup 2^* &= \{x : x \in 1 \divideontimes 2^* \vee x \in 2^* \divideontimes 1\} \\ &= \{x : x \in 1 \vee x \in 2^*\} \\ &= \{\alpha, \beta, 1^*\}, \end{aligned}$$

$$\begin{aligned}
\textit{(d)}\quad \{1\}\cup\{1^*\} &= \{x : x\in\{1\}\circledast\{1^*\}\vee x\in\{1^*\}\circledast\{1\}\}\\
&= \{x : x\in\emptyset\vee x\in\emptyset\}\\
&= \emptyset,
\end{aligned}$$

$$\begin{aligned}
\textit{(e)}\quad \{1\}\cup\{1^*,2^*\} &= \{x : x\in\{1\}\circledast\{1^*,2^*\}\vee x\in\{1^*,2^*\}\circledast\{1\}\}\\
&= \{x : x\in\emptyset\vee x\in\{2^*\}\}\\
&= \{2^*\}.
\end{aligned}$$

Theorem 3.52. *Let X be a σ-set. Then*

(a): $X\cup\emptyset=\emptyset\cup X=X$.
(b): $X\cup X=X$.

Proof.

(a): Since the fusion is commutative, we only prove that $X\cup\emptyset=X$. By Corollary 3.43, we obtain that $X\circledast\emptyset=X$ and $\emptyset\circledast X=\emptyset$. Therefore $X\cup\emptyset=\{x : x\in X\}$. Now, by Axiom 3.2 (Extensionality), $X\cup\emptyset=X$.
(b): It is clear, by Corollary 3.43, that $X\circledast X=X$. Therefore $X\cup X=\{x : x\in X\}$. Then by Axiom 3.2 (Extensionality) $X\cup X=X$.

□

Example 3.53. *Consider the σ-sets $X=\{1,2\}$, $Y=\{1^*,2^*\}$, $Z=\{1\}$ and $W=\{\emptyset\}$. Then we obtain the following:*

(1) $X\cup Y=\emptyset$.
(2) $X\cup W=\{1,2,\emptyset\}$.
(3) $X\cup Z=\{1,2\}$.
(4) $Y\cup Z=\{2^*\}$.

Also we can see that the fusion of σ-sets is not associative. By Theorem 3.52 and Example 3.53 we obtain

$$(Y\cup X)\cup Z=\emptyset\cup Z=Z$$

and

$$Y\cup(X\cup Z)=Y\cup X=\emptyset.$$

It is clear that $Z\neq\emptyset$, then the fusion of σ-sets is not associative. Therefore it is necessary to consider the order on which the σ-sets are founded, thus we introduce the notion of chain of fusion

- $\overrightarrow{F}=X\cup Y\cup Z=(X\cup Y)\cup Z$,
- $\overrightarrow{F}=X\cup Y\cup Z\cup W\cup\ldots=(\ldots(((X\cup Y)\cup Z)\cup W)\cup\ldots)$.

Therefore, given a σ-set F the fusion of all σ-elements of F, or fusion of F, will be denoted for $\bigcup\overrightarrow{F}$.

Then an important question is: for which σ-sets the fusion is associative. Notice that in this case, all chains of fusion are equal.

Theorem 3.54. *If $F \in AF$, then the fusion in F is associative, that is*
$$(F \in AF) \to (\forall X, Y, Z \in F)[(X \cup Y) \cup Z = X \cup (Y \cup Z)].$$

Proof. We consider $F \in AF$ and $X, Y, Z \in F$. Since $F \in AF$, we have that
$$X \widehat{\cap} Y = Y \widehat{\cap} X = X \widehat{\cap} Z = Z \widehat{\cap} X = Y \widehat{\cap} Z = Z \widehat{\cap} Y = \emptyset.$$

Therefore $X \cup Y = \{x : x \in X \vee x \in Y\}$ and $Y \cup Z = \{x : x \in Y \vee x \in Z\}$. This fact implies that
$$(X \cup Y) \widehat{\cap} Z = Z \widehat{\cap} (X \cup Y) = X \widehat{\cap} (Y \cup Z) = (Y \cup Z) \widehat{\cap} X = \emptyset.$$

In fact, suppose that $(X \cup Y) \widehat{\cap} Z \neq \emptyset$. Then there exists $x \in X \cup Y$ such that $x^* \in Z$, which is a contradiction. Now, suppose that $Z \widehat{\cap} (X \cup Y) \neq \emptyset$, then there exists $x \in Z$ such that $x^* \in X \cup Y$ which is a contradiction. The prove that $X \widehat{\cap} (Y \cup Z) = (Y \cup Z) \widehat{\cap} X = \emptyset$ is analogous. Now, by Definition 3.50, we obtain that
$$(X \cup Y) \cup Z = \{x : x \in X \cup Y \vee x \in Z\} = \{x : (x \in X \vee x \in Y) \vee x \in Z\}$$
$$X \cup (Y \cup Z) = \{x : x \in X \vee x \in Y \cup Z\} = \{x : x \in X \vee (x \in Y \vee x \in Z)\}.$$

Finally, by Axiom 3.2 (Extensionality), $(X \cup Y) \cup Z = X \cup (Y \cup Z)$. □

Definition 3.55. *Let X and Y be σ-sets. We say that Y* ***is the σ-antiset of*** *X if the fusion of X and Y is equal to the empty σ-set, that is*
$$X \cup Y = \emptyset.$$

The σ-antiset of X will be denoted by $X^\star$.

Now, consider $X = \{1, 2\}$ then $X^\star = \{1^*, 2^*\}$. We observe that the fusion of pairs of X and $X^\star$ is nonempty because $\min(X) = \{1\}$ and $\min(X^\star) = \{1^*\}$. Therefore $\{X\} \cup \{X^\star\} = \{X, X^\star\}$.

It is clear by Theorem 3.29, that there exist σ-sets without σ-antiset as the case of the σ-set 1_Θ. However, we will prove that, if a σ-set X has a σ-antiset then it is unique.

Lemma 3.56. *Let X and Y be σ-sets. If $X \divideontimes Y = \emptyset$ and $x \in X$, then $x^* \in Y$.*

Proof. Consider X and Y, σ-sets such that $X \divideontimes Y = \emptyset$. By Definition 3.40 we have that $X - (X \widehat{\cap} Y) = \emptyset$. Therefore, if $x \in X$ then $x \in X \widehat{\cap} Y$ and so $x^* \in Y$. □

Lemma 3.57. *Let X and Y be σ-sets. If $X \cup Y = \emptyset$, then*

- **(a):** *If $x \in X$, then $x^* \in Y$.*
- **(b):** *If $y \in Y$, then $y^* \in X$.*

Proof. Suppose that $X \cup Y = \emptyset$. By Definition 3.50 we have that
$$X \cup Y = \{x : (x \in X \divideontimes Y) \vee (x \in Y \divideontimes X)\} = \emptyset.$$

Therefore $X \divideontimes Y = \emptyset$ and $Y \divideontimes X = \emptyset$. Finally by Lemma 3.56 we obtain that if $x \in X$ then $x^* \in Y$ and so if $y \in Y$ then $y^* \in X$. □

Theorem 3.58. *Let X be a σ-set. If there exists $X^\star$ the σ-antiset of X, then $X^\star$ is unique.*

Proof. Suppose that $X = \emptyset$. By Theorem 3.52 there exists $X^\star = \emptyset$ such that $X \cup X^\star = \emptyset$ and it is unique. Now, we consider X, a nonempty σ-set such that there exists $X^\star$, the σ-antiset of X. It is clear, by Theorem 3.52 that $X^\star \neq \emptyset$. Suppose that there exists a σ-set $\widehat{X}$ such that $X \cup \widehat{X} = \emptyset$ and $X^\star \neq \widehat{X}$. Therefore by Theorem 3.52 we have that $\widehat{X} \neq \emptyset$. Since $X^\star \neq \widehat{X}$ then there exists $a \in X^\star$ such that $a \notin \widehat{X}$ or there exists $b \in \widehat{X}$ such that $b \notin X^\star$. Let $a \in X^\star$ and $a \notin \widehat{X}$. Since $a \in X^\star$ then by Lemma 3.57 $a^* \in X$. Therefore $(a^*)^* \in \widehat{X}$. Finally, by Corollary 3.35 we have that $(a^*)^* = a$, which is a contradiction. When there exists $b \in \widehat{X}$ such that $b \notin X^\star$ we get the same contradiction. $\square$

Corollary 3.59. *Let X be a σ-set. If there exists $X^\star$ the σ-antiset of X, then $(X^\star)^\star = X$.*

Proof. This fact is obvious by Theorem 3.58. $\square$

We observe that $\emptyset^\star = \emptyset$ and $(\emptyset^\star)^\star = \emptyset$.

Definition 3.60. *Let X be a σ-set. We define the successor of X by*

$$\begin{aligned} S(X) &= X \cup \{X\}. \\ &= \{x : x \in X \divideontimes \{X\} \vee x \in \{X\} \divideontimes X\} \end{aligned}$$

Lemma 3.61. *Let X be a σ-set. Then the following statements holds:*

- **(a):** $X \divideontimes \{X\} = X$;
- **(b):** $\{X\} \divideontimes X = \{X\}$.

Proof. (a) Suppose that $X = \emptyset$ then it is clear that $\emptyset \divideontimes \{\emptyset\} = \emptyset$. We consider $X \neq \emptyset$, by Definition 3.40, it is clear that $X \divideontimes \{X\} \subseteq X$. Now, suppose that there exists $y \in X$ such that $y \notin X \divideontimes \{X\}$. Since $y \in X$ and $y \notin X - X \widehat{\cap} \{X\}$ then $y \in X \widehat{\cap} \{X\}$. Therefore $y \in y^* = X$ which is a contradiction because $y \not\equiv y^*$.

(b) Suppose that $X = \emptyset$ then it is clear that $\{\emptyset\} \divideontimes \emptyset = \{\emptyset\}$. We consider $X \neq \emptyset$, by Definition 3.40, it is clear that $\{X\} \divideontimes X \subseteq \{X\}$. Now, suppose that $X \notin \{X\} \divideontimes X$. Since $X \in \{X\}$ and $X \notin \{X\} - \{X\} \widehat{\cap} X$ then $X \in \{X\} \widehat{\cap} X$. Therefore $X^* \in X$ which is a contradiction because $X \not\equiv X^*$. $\square$

Corollary 3.62. *Let X be a σ-set. Then $S(X) = \{x : x \in X \vee x \in \{X\}\}$.*

Proof. This fact is obvious by Definition 3.60 and Lemma 3.61 $\square$

Remember that, if X is a σ-set then

$$y \in \min(X) \leftrightarrow (y \in X) \wedge (\forall \triangleleft x, \ldots, w \triangleright \in CH(y))(x, \ldots, w \notin X);$$

$$y \notin \min(X) \leftrightarrow (y \notin X) \vee (\exists \triangleleft x, \ldots, w \triangleright \in CH(y))(x \in X \vee \ldots \vee w \in X).$$

Lemma 3.63. *Let X be a σ-set.*

(a): *If $X = \emptyset$, then* $\min(\emptyset) \subset \min(\{\emptyset\})$.
(b): *If $X \neq \emptyset$ and lower ϵ-bounded, then* $\min(X) = \min(S(X))$.

Proof.
(a) If $X = \emptyset$, then $\min(\emptyset) = \emptyset$. Therefore $\min(\emptyset) \subset \min(\{\emptyset\}) = \{\emptyset\}$.
(b) Now, if $X \neq \emptyset$ and $\min(X) \neq \emptyset$ we obtain the following:

($\subseteq$): Suppose that there exists $y \in \min(X)$ such that $y \notin \min(S(X))$. By Corollary 3.62 it is clear that $y \in S(X)$, because $y \in \min(X) \subseteq X \subset S(X)$. Since, $y \in \min(X)$ we have that for all $\triangleleft x, \ldots, w \triangleright \in CH(y)$ then $x, \ldots, w \notin X$. Also, as $y \in S(X)$ and $y \notin \min(S(X))$, there exists $\triangleleft a, \ldots, c \triangleright \in CH(y)$ such that $a \in S(X) - X \vee \ldots \vee c \in S(X) - X$. Finally there exists $\triangleleft a, \ldots, c, y \triangleright \in CH(X)$ such that $X \in \triangleleft a, \ldots, c, y \triangleright$ which contradicts Axiom 3.6 (Weak Regularity).

($\supseteq$): Let $y \in \min(S(X))$. It is clear that $y \in X$ because $X \notin \min(S(X))$. Now, we consider and $\triangleleft x, \ldots, w \triangleright \in CH(y)$. Since $y \in \min(S(X))$ we have that $x, \ldots, w \notin S(X)$. Therefore, by Corollary 3.62 we obtain that $x, \ldots, w \notin X$. So $y \in \min(X)$.

$\square$

Lemma 3.64. *Let X and Y be nonempty σ-sets. If $X \not\equiv Y$, then $S(X) \not\equiv S(Y)$.*

Proof. We consider X, Y nonempty σ-sets such that $X \not\equiv Y$. That is

$$(\forall \triangleleft x \ldots w \triangleright \in CH(X))(\forall \triangleleft a \ldots c \triangleright \in CH(Y))(\triangleleft x \ldots w \triangleright \not\equiv \triangleleft a \ldots c \triangleright).$$

Now, let $\triangleleft x \ldots w \triangleright \in CH(S(X))$ and $\triangleleft a \ldots c \triangleright \in CH(S(Y))$. By Corollary 3.62, the respective greater links w and c satisfy the following conditions:

(1) $w \in X$ or $w = X$.
(2) $c \in Y$ or $c = Y$.

Finally, it is clear that

- If $w \in X$ and $c \in Y$, then $\triangleleft x \ldots w \triangleright \not\equiv \triangleleft a \ldots c \triangleright$.
- If $w \in X$ and $c = Y$, then $\triangleleft x \ldots w \triangleright \not\equiv \triangleleft a \ldots c \triangleright$.
- If $w = X$ and $c \in Y$, then $\triangleleft x \ldots w \triangleright \not\equiv \triangleleft a \ldots c \triangleright$.
- If $w = X$ and $c = Y$, then $\triangleleft x \ldots w \triangleright \not\equiv \triangleleft a \ldots c \triangleright$.

because $X \not\equiv Y$. Therefore $S(X) \not\equiv S(Y)$. $\square$

Now, we introduce the following schemas,

$$\begin{array}{ccccccccccccc}
\cdots & \to & \alpha & \to & 1 & \to & 2 & \to & 3 & \to & 4 & \to & \cdots \\
 & & & \searrow & & & & & & & & & \\
 & & & & 1_\Lambda & \to & 2_\Lambda & \to & 3_\Lambda & \to & 4_\Lambda & \to & \cdots \\
 & & & \nearrow & & & & & & & & & \\
 & & \emptyset & \to & 1_\Theta & \to & 2_\Theta & \to & 3_\Theta & \to & 4_\Theta & \to & \cdots \\
 & & & \searrow & & & & & & & & & \\
 & & & & 1_\Omega & \to & 2_\Omega & \to & 3_\Omega & \to & 4_\Omega & \to & \cdots \\
 & & & \nearrow & & & & & & & & & \\
\cdots & \to & \beta & \to & 1^* & \to & 2^* & \to & 3^* & \to & 4^* & \to & \cdots
\end{array}$$

$$\begin{array}{ccccccccccccc}
\cdots & \to & \alpha & & & & & & & & & & \\
 & & & \searrow & & & & & & & & & \\
 & & & & 1_\Gamma & \to & 2_\Gamma & \to & 3_\Gamma & \to & 4_\Gamma & \to & \cdots \\
 & & & \nearrow & & & & & & & & & \\
\cdots & \to & \beta & & & & & & & & & &
\end{array}$$

$$\begin{array}{ccccccccccccc}
\cdots & \to & \alpha & & & & & & & & & & \\
 & & & \searrow & & & & & & & & & \\
 & & \emptyset & \to & 1_\Pi & \to & 2_\Pi & \to & 3_\Pi & \to & 4_\Pi & \to & \cdots \\
 & & & \nearrow & & & & & & & & & \\
\cdots & \to & \beta & & & & & & & & & &
\end{array}$$

where we have that

- $1 = \{\alpha\}$, $2 = \{\alpha, 1\}$, $3 = \{\alpha, 1, 2\}$, $4 = \{\alpha, 1, 2, 3\}$, ...
- $1_\Pi = \{\emptyset, \alpha, \beta\}$, $2_\Pi = \{\emptyset, \alpha, \beta, 1_\Pi\}$, $3_\Pi = \{\emptyset, \alpha, \beta, 1_\Pi, 2_\Pi\}$, ...
- $1_\Lambda = \{\emptyset, \alpha\}$, $2_\Lambda = \{\emptyset, \alpha, 1_\Lambda\}$, $3_\Lambda = \{\emptyset, \alpha, 1_\Lambda, 2_\Lambda\}$, ...
- $1_\Theta = \{\emptyset\}$, $2_\Theta = \{\emptyset, 1_\Theta\}$, $3_\Theta = \{\emptyset, 1_\Theta, 2_\Theta\}$, ...
- $1_\Omega = \{\emptyset, \beta\}$, $2_\Omega = \{\emptyset, \beta, 1_\Omega\}$, $3_\Omega = \{\emptyset, \beta, 1_\Omega, 2_\Omega\}$, ...
- $1_\Gamma = \{\alpha, \beta\}$, $2_\Gamma = \{\alpha, \beta, 1_\Gamma\}$, $3_\Gamma = \{\alpha, \beta, 1_\Gamma, 2_\Gamma\}$, ...
- $1^* = \{\beta\}$, $2^* = \{\beta, 1^*\}$, $3^* = \{\beta, 1^*, 2^*\}$, $4^* = \{\beta, 1^*, 2^*, 3^*\}$, ...

We can see these σ-sets related in the following schema:

$$
\begin{array}{ccccccccc}
 & & & & 1_\Theta & & & & \\
 & & & & \uparrow & & & & \\
 & & & & \emptyset & & & & \\
 & & & \swarrow & \downarrow & \searrow & & & \\
 & & 1_\Lambda & & 1_\Pi & & 1_\Omega & & \\
 & & \uparrow & \nearrow & & \nwarrow & \uparrow & & \\
 & & \alpha & & & & \beta & & \\
 & \swarrow & & \searrow & & \swarrow & & \searrow & \\
1 & & & & 1_\Gamma & & & & 1^*
\end{array}
$$

Therefore, we can introduce the following definition:

Definition 3.65. *Let I be a σ-set,*

(a): *I is called inductive if:*
(1) $\min(I) \neq \emptyset$;
(2) *If $x \in I$, then $S(x) \in I$.*

$$
\begin{array}{ccccccccc}
1 & \in & 2 & \in & 3 & \in & 4 & \in & \cdots \\
1_\Lambda & \in & 2_\Lambda & \in & 3_\Lambda & \in & 4_\Lambda & \in & \cdots \\
1_\Theta & \in & 2_\Theta & \in & 3_\Theta & \in & 4_\Theta & \in & \cdots
\end{array}
$$

(b): *I is called α-inductive if:*
(1) $\min(I) = \{1\}$;
(2) *If $x \in I$ and $x \neq 1$, then $1 \in x$.*
(3) *If $x \in I$, then $S(x) \in I$.*
In this case we denote the α-inductive σ-set by I_α.

$$
1 \in 2 \in 3 \in 4 \in \cdots
$$

(c): *I is called β-inductive if:*
(1) $\min(I) = \{1^*\}$;
(2) *If $x \in I$ and $x \neq 1^*$, then $1^* \in x$.*
(3) *If $x \in I$, then $S(x) \in I$.*
In this case we denote the β-inductive σ-set by I_β.

$$
1^* \in 2^* \in 3^* \in 4^* \in \cdots
$$

(d): *I is called Θ-inductive if:*
(1) $\min(I) = \{1_\Theta\}$;
(2) *If $x \in I$ and $x \neq 1_\Theta$, then $1_\Theta \in x$.*
(3) *If $x \in I$, then $S(x) \in I$.*
In this case we denote the Θ-inductive σ-set by I_Θ.

$$1_\Theta \in 2_\Theta \in 3_\Theta \in 4_\Theta \in \cdots$$

We observe that if I is Θ-inductive, α-inductive or β-inductive, then it is inductive.

Remember that, if X is a σ-set then

$$y \in \max(X) \leftrightarrow (y \in X) \wedge (\forall z \in X)(\forall \lhd x, \ldots, w \rhd \in CH(z))(y \notin \lhd x, \ldots, w \rhd);$$

$$y \notin \max(X) \leftrightarrow (y \notin X) \vee (\exists z \in X)(\exists \lhd x, \ldots, w \rhd \in CH(z))(y \in \lhd x, \ldots, w \rhd).$$

Theorem 3.66. *Let I be an inductive σ-set. Then I is a non upper ϵ-bounded σ-set.*

Proof. It is clear that I is lower ϵ-bounded by Definition 3.65. Now, suppose that there exists $y \in \max(I)$. Therefore, for all $z \in I$ and for all $\lhd x, \ldots, w \rhd \in CH(z)$ we have that $y \notin \lhd x, \ldots, w \rhd$.

On the other hand, since $y \in I$ and I is an inductive σ-set, then $S(y) \in I$. Therefore there exists $S(y) \in I$ such that $y \in S(y)$ which is a contradiction. So $\max(I) = \emptyset$. □

The existence of inductive σ-sets is guaranteed by Axiom 3.7 (non ϵ-Bounded σ-Set).

Now, we introduce the concept of σ-set of all natural numbers, anti-natural numbers and Θ-natural numbers.

Definition 3.67. *Let X_α, X_β and X_Θ be α, β or Θ-inductive σ-sets. Then we define:*

(a): *The σ-set of all natural numbers as:*

$$IN = \{x \in X_\alpha : (x \in I_\alpha)(\forall I_\alpha)\}.$$

(b): *The σ-set of all anti-natural numbers as:*

$$IN^\star = \{x \in X_\beta : (x \in I_\beta)(\forall I_\beta)\}.$$

(c): *The σ-set of all Θ-natural numbers as:*

$$IN_\Theta = \{x \in X_\Theta : (x \in I_\Theta)(\forall I_\Theta)\}.$$

Theorem 3.68. *The following statements holds:*

(a): *IN is α-inductive and if I_α is any α-inductive σ-set, then $IN \subseteq I_\alpha$;*
(b): *$IN^\star$ is β-inductive and if I_β is any β-inductive σ-set, then $IN^\star \subseteq I_\beta$;*
(c): *IN_Θ is Θ-inductive and if I_Θ is any Θ-inductive σ-set, then $IN \subseteq I_\Theta$.*

Proof. We will only prove (a) because the proofs of (b) and (c) are similar.

- If $x \in \mathrm{IN}$, then $x \in I_\alpha$ for all I_α. Therefore if $x \neq 1$ then $1 \in x$.
- If $x \in \mathrm{IN}$, then $x \in I_\alpha$ for all I_α, so $S(x) \in I_\alpha$ for all I_α. Therefore $S(x) \in \mathrm{IN}$.

- It is clear that $1 \in \mathrm{IN}$ because $1 \in I_\alpha$ for any I_α. Now we will prove that $\min(\mathrm{IN}) = \{1\}$. It is clear by definition that

  $$y \in \min(I) \leftrightarrow (y \in I) \wedge (\forall \triangleleft x, \ldots, w \triangleright \in CH(y))(x, \ldots, w \notin I).$$

  Therefore we obtain that $1 \in \min(\mathrm{IN})$ because $1 \in \mathrm{IN}$ and for all $\triangleleft x, \ldots, w \triangleright \in CH(1)$ we have that $x, \ldots, w \notin I_\alpha$ for all I_α. So $x, \ldots, w \notin \mathrm{IN}$. Hence $\{1\} \subseteq \min(\mathrm{IN})$.

  Now, suppose that $\min(\mathrm{IN}) \not\subseteq \{1\}$. Then there exists $y \in \min(\mathrm{IN})$ such that $y \neq 1$. Since $y \in \min(\mathrm{IN})$ then for all $\triangleleft x, \ldots, w \triangleright \in CH(y)$ we have that $x, \ldots, w \notin \mathrm{IN}$. Therefore, we obtain $y \in \mathrm{IN}$ such that $y \neq 1$ and for all $\triangleleft x, \ldots, w \triangleright \in CH(y)$ we have that $x, \ldots, w \notin \mathrm{IN}$, which is a contradiction because $1 \in y$ and $1 \in \mathrm{IN}$. Therefore $\min(\mathrm{IN}) = \{1\}$.

The second part of the Theorem 3.68 (a) follows immediately from the definition of IN. □

Now, we introduce the Principle of Induction for the study of the different types of natural numbers IN, $\mathrm{IN}^\star$ and IN_Θ.

Theorem 3.69. ***The Principle of Induction.*** *Let $\Phi(x)$ be a normal formula.*

(1) $[\Phi(1) \wedge (\forall n \in IN)(\Phi(n) \to \Phi(n \cup \{n\}))] \to (\forall n \in IN)(\Phi(n))$.
(2) $[\Phi(1^*) \wedge (\forall n \in IN^\star)(\Phi(n) \to \Phi(n \cup \{n\}))] \to (\forall n \in IN^\star)(\Phi(n))$.
(3) $[\Phi(1_\Theta) \wedge (\forall n \in IN_\Theta)(\Phi(n) \to \Phi(n \cup \{n\}))] \to (\forall n \in IN_\Theta)(\Phi(n))$.

Proof. We will only prove (a) because the proofs of (b) and (c) are similar. Let $\Phi(x)$ a normal formula and $A = \{n \in \mathrm{IN} : \Phi(n)\}$. By Theorem 3.6 it is clear that A is a σ-set. Now, we prove that A is α-inductive.

If $n \in A$ then $n \in \mathrm{IN}$. Therefore if $n \neq 1$ then $1 \in n$.

If $n \in A$ then $\Phi(n)$ holds. So $S(n) \in \mathrm{IN}$ and $\Phi(S(n))$ holds, which implies that $S(n) \in A$.

It is clear that $A \subseteq \mathrm{IN}$ and $1 \in A$. Let $\triangleleft x, \ldots, w \triangleright \in CH(1)$, then $x, \ldots, w \notin \mathrm{IN}$ because $1 \in \min(\mathrm{IN})$. Therefore $x, \ldots, w \notin A$ and so $1 \in \min(A)$. Now, suppose that $\min(A) \not\subseteq \{1\}$. Then there exists $y \in \min(A)$ such that $y \neq 1$. So, we obtain a σ-element $y \in A$ such that $y \neq 1$. Hence $1 \in y$, which is a contradiction. Therefore $\min(A) = \{1\}$ and so A is α-inductive. Finally $\mathrm{IN} = A$. □

Lemma 3.70. *The following statements holds:*

(a): *For all $n \in IN$,* $\min(n) = 1$,
(b): *For all $n \in IN^\star$,* $\min(n) = 1^*$,
(c): *For all $n \in IN_\Theta$,* $\min(n) = 1_\Theta$.

Proof. We will only prove (a) because the proofs of (b) and (c) are similar. It is clear that $\min(1) = 1$. Suppose that $\min(n) = 1$, then, by Lemma

3.63, we have that $\min(S(n)) = 1$. Finally, by Theorem 3.69, we obtain that $\min(n) = 1$ for all $n \in \mathrm{IN}$. $\square$

Theorem 3.71. *For all $n \in IN$ there exists a unique σ-set m such that $\{n\} \cup \{m\} = \emptyset$.*

Proof. If $n = 1$ then, by Theorem 3.36, there exists a unique σ-set 1^* such that $\{1\} \cup \{1^*\} = \emptyset$. Suppose that given $n \in \mathrm{IN}$ there exists a unique σ-set m such that $\{n\} \cup \{m\} = \emptyset$. Now, we will prove that for $S(n)$ there exists a unique σ-set $\widehat{m}$ such that $\{S(n)\} \cup \{\widehat{m}\} = \emptyset$.

Existence: Consider $\widehat{m} = S(m)$. By Lemmas 3.64 and 3.70, we have that conditions (a) and (b) of Axiom 3.12 (Completeness B) are satisfied by $S(n)$ and $S(m)$. Now, we only need to prove that conditions (c) and (d) of Axiom 3.12 are satisfied by them. Nevertheless, we only prove condition (c) because the proof of condition (d) is analogous.

Let $z \in S(n)$ such that $z \notin \min(S(n)) = \min(n) = 1$. By Corollary 3.62, we have that $z \in n$ or $z = n$.

- **(case 1):** Suppose that $z \in n$.
  - **(a.1):** Since $z \in n$ and $z \notin \min(S(n)) = \min(n)$, then we have that there exists a unique w such that $\{z\} \cup \{w\} = \emptyset$, because n and m satisfy the condition (c) of Axiom 3.12. Also, we observe that $w \in m$ for the same condition.
  - **(b.1):** Consider a such that $\{z\} \cup \{a\} = \emptyset$. By (a.1) we obtain that $a = w$. Therefore $a \in m$ and by Corollary 3.62 we have that $a \in S(m)$.
- **(case 2):** Suppose that $z = n$.
  - **(a.2):** Since $z = n$ then it is clear, by inductive hypothesis, that there exists a unique m such that $\{n\} \cup \{m\} = \emptyset$.
  - **(b.2):** Consider a such that $\{n\} \cup \{a\} = \emptyset$. By (a.2) we obtain that $a = m$. Finally by Corollary 3.62 we have that $a \in S(m)$.

Therefore we have that

$$\{S(n)\} \cup \{S(m)\} = \emptyset.$$

Uniqueness: This fact is obvious by Theorem 3.34. $\square$

Theorem 3.72. *For all $n \in IN^\star$ there exists a unique σ-set m such that $\{n\} \cup \{m\} = \emptyset$.*

Proof. This proof is analogous to the previous one. $\square$

Now, we include the following Corollary in order to calculate the σ-anti-elements of the σ-elements of IN and $\mathrm{IN}^\star$.

Corollary 3.73. *The following statements holds:*

- **(a):** *For all $n \in IN$, $S(n^*) = S(n)^*$.*
- **(b):** *For all $n \in IN^\star$, $S(n^*) = S(n)^*$.*

Proof. We will only prove (a) because the proof of (b) is analogous.

Consider $n = 1$. Then by Theorem 3.37 we obtain that $S(1^*) = S(1)^* = 2^*$. Let $n \in$ IN such that $S(n^*) = S(n)^*$. Now, we will prove that $S(S(n)^*) = S(S(n))^*$. Since $S(n^*) = S(n)^*$ then

$$S(S(n)^*) = S(S(n))^* \leftrightarrow S(S(n^*)) = S(S(n))^*.$$

Therefore by Theorem 3.71 it is only necessary to prove that

$$\{S(S(n))\} \cup \{S(S(n^*))\} = \emptyset.$$

Since $\{S(n)\} \cup \{S(n^*)\} = \emptyset$ then by Lemmas 3.64 and 3.70, we obtain that $S(S(n))$ and $S(S(n^*))$ satisfy the conditions (a) and (b) of Axiom 3.12 (Completeness B). Now, we only need to prove that conditions (c) and (d) of Axiom 3.12 are satisfied by them. Nevertheless, we only prove condition (c) because the proof of condition (d) is analogous.

Let $z \in S(S(n))$ such that $z \notin \min(S(S(n))) = \min(S(n)) = 1$. By Corollary 3.62 we have that $z \in S(n)$ or $z = S(n)$.

- **(case 1):** Suppose that $z \in S(n)$.
  - **(a.1):** Since $z \in S(n)$ and $z \notin \min(S(S(n))) = \min(S(n))$, then we have that there exists a unique w such that $\{z\} \cup \{w\} = \emptyset$ because $S(n)$ and $S(n^*)$ satisfy the condition (c) of Axiom 3.12. Also, we observe that $w \in S(n^*)$, for the same condition.
  - **(b.1):** Consider a such that $\{z\} \cup \{a\} = \emptyset$. By (a.1) we obtain that $a = w$. Therefore $a \in S(n^*)$ and by Corollary 3.62 we have that $a \in S(S(n^*))$.
- **(case 2):** Suppose that $z = S(n)$.
  - **(a.2):** Since $z = S(n)$ then it is clear, by inductive hypothesis, that there exists a unique $S(n^*)$ such that $\{S(n)\} \cup \{S(n^*)\} = \emptyset$.
  - **(b.2):** Consider a such that $\{S(n)\} \cup \{a\} = \emptyset$. By (a.2) we obtain that $a = S(n^*)$. Finally by Corollary 3.62 we have that $a \in S(S(n^*))$.

Therefore we have that

$$\{S(S(n))\} \cup \{S(S(n^*))\} = \emptyset.$$

□

Theorem 3.74. *The following statements holds:*

- **(a):** *For all $n \in IN$, $n^* \in IN^\star$.*
- **(b):** *For all $n \in IN^\star$, $n^* \in IN$.*

Proof. We will only prove (a) because the proof of (b) is analogous. If $n = 1$ then it is clear that $(1)^* = 1^* \in \mathrm{IN}^\star$. Let $n \in$ IN such that $n^* \in \mathrm{IN}^\star$. Since IN and $\mathrm{IN}^\star$ are inductive σ-sets then $S(n) \in$ IN and $S(n^*) \in \mathrm{IN}^\star$. Finally, by Corollary 3.73 we have that $S(n^*) = S(n)^* \in \mathrm{IN}^\star$. □

We will use the following notation:

- n, m, i, j, k, etc. to denote natural numbers.
- n^*, m^*, i^*, j^*, k^*, etc. to denote anti-natural numbers.
- $n_\Theta, m_\Theta, i_\Theta, j_\Theta, k_\Theta$, etc. to denote Θ-natural numbers.
- If $n \in \mathrm{IN}$ then $S(n) = n + 1 \in \mathrm{IN}$,
$$\mathrm{IN} = \{1, 2, 3, 4, \ldots\},$$
1, $S(1) = 1 + 1 = 2$, $S(2) = 2 + 1 = 3$, $S(3) = 3 + 1 = 4$, etc.
- If $n^* \in \mathrm{IN}^\star$ then $S(n^*) = n^* + 1^* \in \mathrm{IN}^\star$,
$$\mathrm{IN}^\star = \{1^*, 2^*, 3^*, 4^*, \ldots\},$$
1^*, $S(1^*) = 1^* + 1^* = 2^*$, $S(2^*) = 2^* + 1^* = 3^*$, $S(3^*) = 3^* + 1^* = 4^*$, etc.
- If $n_\Theta \in \mathrm{IN}_\Theta$ then $S(n_\Theta) = n_\Theta + 1_\Theta \in \mathrm{IN}_\Theta$,
$$\mathrm{IN}_\Theta = \{1_\Theta, 2_\Theta, 3_\Theta, 4_\Theta, \ldots\},$$
$1_\Theta, S(1_\Theta) = 1_\Theta + 1_\Theta = 2_\Theta, S(2_\Theta) = 2_\Theta + 1_\Theta = 3_\Theta, S(3_\Theta) = 3_\Theta + 1_\Theta = 4_\Theta$, etc.

Therefore, by Corollary 3.73, we obtain that
$$\{1\} \cup \{1^*\} = \{2\} \cup \{2^*\} = \{3\} \cup \{3^*\} = \{4\} \cup \{4^*\} = \ldots = \emptyset.$$

3.16. **The Axiom of Generated σ-set.** For all σ-sets X and Y there exists a σ-set, called the σ-set generated by X and Y, whose σ-elements are exactly the fusion of the σ-subsets of X with the σ-subsets of Y, that is
$$(\forall X, Y)(\exists Z)(\forall a)(a \in Z \leftrightarrow (\exists A \in 2^X)(\exists B \in 2^Y)(a = A \cup B)).$$

Now we can define the Generated Space by X and Y.

Definition 3.75. *Let X and Y be σ-sets. The **Generated Space by X and Y** is given by*
$$\langle 2^X, 2^Y \rangle = \{A \cup B : A \in 2^X \wedge B \in 2^Y\}.$$

We observe that in general
$$2^{X \cup Y} \neq \langle 2^X, 2^Y \rangle.$$

Consider $X = \{1, 2^*\}$ and $Y = \{1, 2\}$, then $X \cup Y = \{1\}$. Therefore $2^{X \cup Y} = \{\emptyset, \{1\}\}$ and $\langle 2^X, 2^Y \rangle = \{\emptyset, \{1\}, \{2\}, \{2^*\}, \{1, 2\}, \{1, 2^*\}\}$.

Now, we consider $X = \{1_\Theta\}$ and $Y = \{1, 2\}$, then we obtain that
$$\begin{aligned}\langle 2^{X \cup Y}, 2^{X \cup Y^\star} \rangle &= \{\emptyset, \{1_\Theta\}, \{1\}, \{2\}, \{1^*\}, \{2^*\}, \{1_\Theta, 1\}, \{1_\Theta, 2\} \\ &\quad \{1_\Theta, 1^*\}, \{1_\Theta, 2^*\}, \{1^*, 2\}, \{1, 2^*\}, \{1, 2\}, \{1^*, 2^*\}, \\ &\quad \{1_\Theta, 1^*, 2\}, \{1_\Theta, 1, 2^*\}, \{1_\Theta, 1, 2\}, \{1_\Theta, 1^*, 2^*\}\}.\end{aligned}$$

Therefore we can build the following schema,

$$
\begin{array}{ccccccccc}
 & & \{1_\Theta, 1^*\} & \rightarrow & \{1_\Theta, 1^*, 2\} & \leftarrow & \{1_\Theta, 2\} & & \\
 & \swarrow & & \nwarrow & \uparrow & \nearrow & & \searrow & \\
\{1_\Theta, 1^*, 2^*\} & & \longleftarrow & & \{1_\Theta, \} & & \longrightarrow & & \{1_\Theta, 1, 2\} \\
 & \nwarrow & & \swarrow & \downarrow & \searrow & & \nearrow & \\
 & & \{1_\Theta, 2^*\} & \rightarrow & \{1_\Theta, 1, 2^*\} & \leftarrow & \{1_\Theta, 1\} & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \{2^*\} & \rightarrow & \{1, 2^*\} & \leftarrow & \{1\} & & \\
 & \swarrow & & \nwarrow & \uparrow & \nearrow & & \searrow & \\
\{1^*, 2^*\} & & \longleftarrow & & \emptyset & & \longrightarrow & & \{\ 1, 2\ \} \\
 & \nwarrow & & \swarrow & \downarrow & \searrow & & \nearrow & \\
 & & \{1^*\} & \rightarrow & \{1^*, 2\} & \leftarrow & \{2\} & &
\end{array}
$$

where $x \rightarrow y := x \subseteq y$.

Definition 3.76. *Let X and Y be σ-sets. We say that $\langle 2^X, 2^Y \rangle$ is the Integer Space generated by X if Y is the σ-antiset of X (i.e. $X^\star = Y$). In this case the Integer Space is denoted by*

$$3^X = \langle 2^X, 2^{X^\star} \rangle.$$

Example 3.77. *(a) Consider $X = \{1, 2\}$ and $X^\star = \{1^*, 2^*\}$, then*

$$3^X = \{\emptyset, \{1\}, \{2\}, \{1^*\}, \{2^*\}, \{1, 2^*\}, \{1^*, 2\}, \{1^*, 2^*\}, \{1, 2\}\}.$$

Therefore we obtain the following schema

$$
\begin{array}{ccccccccc}
 & & & & \{1, 2^*\} & & & & \\
 & & & \nearrow & & \nwarrow & & & \\
 & & \{2^*\} & & \uparrow & & \{1\} & & \\
 & \swarrow & & \nwarrow & & \nearrow & & \searrow & \\
\{1^*, 2^*\} & & \longleftarrow & & \emptyset & & \longrightarrow & & \{\ 1, 2\ \} \\
 & \nwarrow & & \swarrow & & \searrow & & \nearrow & \\
 & & \{1^*\} & & \downarrow & & \{2\} & & \\
 & & & \searrow & & \swarrow & & & \\
 & & & & \{1^*, 2\} & & & &
\end{array}
$$

(b) Consider $X = \{1, 2, 3\}$ and $X^\star = \{1^*, 2^*, 3^*\}$, then

$3^X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1^*\}, \{2^*\}, \{3^*\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1^*, 2\}, \{1^*, 3\}, \{2^*, 3\}, \{1^*, 2^*\}, \{1^*, 3^*\}, \{2^*, 3^*\}, \{1, 2^*\}, \{1, 3^*\}, \{2, 3^*\}, \{1, 2, 3\}, \{1^*, 2, 3\}, \{1, 2^*, 3\}, \{1, 2, 3^*\}, \{1^*, 2^*, 3\}, \{1^*, 2, 3^*\}, \{1, 2^*, 3^*\}, \{1^*, 2^*, 3^*\}\}.$

4. Final Comments.

The study of axiomatic system of σ-Set Theory we are leads, among other things, to the build of σ-sets IN, $IN^\star$ and IN_Θ. Now, it is important to observe some of the basic properties of these σ-sets.

By definition we have that:

- $IN = \{1, 2, 3, 4, 5, \ldots\}$;
- $IN^\star = \{1^*, 2^*, 3^*, 4^*, 5^*, \ldots\}$;
- $IN_\Theta = \{1_\Theta, 2_\Theta, 3_\Theta, 4_\Theta, 5_\Theta, \ldots\}$.

Now, if we consider $n, m \in IN$ such that $n \neq m$, then we have that

- $\{n\} \cup \{n^*\} = \{n^*\} \cup \{n\} = \emptyset$;
- $\{n\} \cup \{m^*\} = \{m^*\} \cup \{n\} = \{n, m^*\}$;
- $\{n\} \cup \{n_\Theta\} = \{n_\Theta\} \cup \{n\} = \{n, n_\Theta\}$;
- $\{n^*\} \cup \{n_\Theta\} = \{n_\Theta\} \cup \{n^*\} = \{n^*, n_\Theta\}$;
- $\{n\} \cup \{m_\Theta\} = \{m_\Theta\} \cup \{n\} = \{n, m_\Theta\}$;
- $\{n^*\} \cup \{m_\Theta\} = \{m_\Theta\} \cup \{n^*\} = \{n^*, m_\Theta\}$.

Hence, a direct consequence of these properties is that

- $IN \cup IN^\star = \emptyset$;
- $IN \cup IN_\Theta = \{1, 1_\Theta, 2, 2_\Theta, 3, 3_\Theta, 4, 4_\Theta, 5, 5_\Theta, \ldots\}$;
- $IN^\star \cup IN_\Theta = \{1^*, 1_\Theta, 2^*, 2_\Theta, 3^*, 3_\Theta, 4^*, 4_\Theta, 5^*, 5_\Theta, \ldots\}$.

In this paper, we give only the mathematical foundations of σ-Set Theory, thus, in the case that the axiomatic system will be consistent, in future work we will see the scope and limitations of the theory. However, we think that all definitions and theorems which are studied in a standard Set Theory, can be built on the σ-Set Theory. On the other hand, it is clear that the reciprocal is false because in a standard set theory, a set does not have the inverse element for the union operation.

Also, we think that the study of the properties of space generated by two σ-sets, as the study of power set in standard set theory, will lead to interesting results about the cardinality of a σ-set. Thus, we will can establish new relationships and properties of infinite cardinals. Now, we present the following conjecture: If X is a σ-set and $|X|$ is the cardinal of X then we have that

Conjecture 4.1. *For all $X \in 2^{IN_\Theta}$ and $Y \in 2^{IN}$, then*

$$|\langle 2^{X\cup Y}, 2^{X\cup Y^\star}\rangle| = 2^{|X|}3^{|Y|}.$$

Therefore, if $X = \emptyset$, then $|\langle 2^{X\cup Y}, 2^{X\cup Y^\star}\rangle| = |3^Y| = 3^{|Y|}$. On the other hand, if $Y = \emptyset$, then $|\langle 2^{X\cup Y}, 2^{X\cup Y^\star}\rangle| = |2^X| = 2^{|X|}$, because $\emptyset = \emptyset^\star$.

Example 4.2. *We consider $X = \{1_\Theta\}$ and $Y = \{1, 2\}$, then we obtain that*

$$\begin{aligned} \langle 2^{X\cup Y}, 2^{X\cup Y^\star}\rangle \;&= \{\emptyset, \{1_\Theta\}, \{1\}, \{2\}, \{1^*\}, \{2^*\}, \{1_\Theta, 1\}, \{1_\Theta, 2\} \\ &\quad \{1_\Theta, 1^*\}, \{1_\Theta, 2^*\}, \{1^*, 2\}, \{1, 2^*\}, \{1, 2\}, \{1^*, 2^*\}, \\ &\quad \{1_\Theta, 1^*, 2\}, \{1_\Theta, 1, 2^*\}, \{1_\Theta, 1, 2\}, \{1_\Theta, 1^*, 2^*\}\}. \end{aligned}$$

So, we have that $|X| = 1$, $|Y| = 2$ *and* $|\langle 2^{X\cup Y}, 2^{X\cup Y^{\star}}\rangle| = 2^1 \cdot 3^2 = 18$.

Example 4.3. *We consider* $X = \emptyset$ *and* $Y = \{1, 2\}$*, then we obtain that*

$$3^Y = \{\emptyset, \{1\}, \{2\}, \{1^*\}, \{2^*\}, \{1^*, 2\}, \{1, 2^*\}, \{1, 2\}, \{1^*, 2^*\}\}.$$

Hence, we have that $|X| = 0$, $|Y| = 2$ *and* $|\langle 2^{X\cup Y}, 2^{X\cup Y^{\star}}\rangle| = 2^0 \cdot 3^2 = 9$.

Respect to the algebraic structure of the Integer Space 3^X and generated space $\langle 2^X, 2^Y\rangle$, we think that these structures are related with the structures called NAFIL (non-associative finite invertible loops). Now, we present our conjecture. If we define

$$A \oplus B \leftrightarrow A \cup B,$$

then we obtain the following.

Conjecture 4.4. *For all* $X \in 2^{IN}$ *we have that* $(3^X, \oplus)$ *satisfies the following conditions:*

(1) $(\forall A, B \in 3^X)(A \oplus B \in 3^X)$.
(2) $(\exists! \xi \in 3^X)(\forall A \in 3^X)(\xi \oplus A = A \oplus \xi = A)$.
(3) $(\forall A \in 3^X)(\exists! A^{\star} \in 3^X)(A \oplus A^{\star} = A^{\star} \oplus A = \xi)$.
(4) $(\forall A, B \in 3^X)(A \oplus B = B \oplus A)$.

It is clear, by Example 4.3 that 3^Y satisfies the conditions (1),(2),(3) and (4) from the Conjecture 4.4.

Now, following with the possible applications of the σ-Set Theory and taking into account the developments of Lattices, we define the following relation: If we consider $A, B \in 3^X$ then we define

$$A \leq B \leftrightarrow B \oplus A^{\star} \in 2^X.$$

Thus, we obtain the following conjecture:

Conjecture 4.5. *For all* $X \in 2^{IN}$ *we have that* $(3^X, \leq)$ *satisfies the following conditions:*

(1) $(\forall A \in 3^X)(A \leq A)$.
(2) $(\forall A, B \in 3^X)(A \leq B \wedge B \leq A \rightarrow A = B)$.
(3) $(\forall A, B \in 3^X)(A \leq B \wedge B \leq C \rightarrow A \leq C)$.

Therefore, for all $X \in 2^{IN}$ we have that $(3^X, \leq)$ is a **partially ordered** σ**-set** and 2^X represent the **positive cone** of 3^X. That is

$$2^X = \{x \in 3^X : x \geq \emptyset\}.$$

Example 4.6. *We consider* $X = \{1, 2, 3\}$ *and the Integer Space* 3^X *(see example 3.77). Now, if we consider the relation*

$$A \leq B \leftrightarrow B \oplus A^{\star} \in 2^X,$$

where $2^X = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$*, then we have that*

$$2^X = \{x \in 3^X : x \geq \emptyset\}.$$

Hence, $(3^X, \leq)$ satisfies the conditions (1),(2) and (3) from the Conjecture 4.5. Therefore, we have that

$$\{1^*,2^*,3^*\} \leq \{1^*,2^*\} \leq \{1^*\} \leq \emptyset \leq \{1\} \leq \{1,2\} \leq \{1,2,3\},$$

$$\{1^*,2^*,3^*\} \leq \{1^*,3^*\} \leq \{1^*\} \leq \emptyset \leq \{1\} \leq \{1,3\} \leq \{1,2,3\},$$

$$\{1^*,2^*,3^*\} \leq \{1^*,2^*\} \leq \{2^*\} \leq \emptyset \leq \{2\} \leq \{1,2\} \leq \{1,2,3\},$$

$$\{1^*,2^*,3^*\} \leq \{2^*,3^*\} \leq \{2^*\} \leq \emptyset \leq \{2\} \leq \{2,3\} \leq \{1,2,3\},$$

$$\{1^*,2^*,3^*\} \leq \{1^*,3^*\} \leq \{3^*\} \leq \emptyset \leq \{3\} \leq \{1,3\} \leq \{1,2,3\},$$

$$\{1^*,2^*,3^*\} \leq \{2^*,3^*\} \leq \{3^*\} \leq \emptyset \leq \{3\} \leq \{2,3\} \leq \{1,2,3\}.$$

Also, we observe, for example, that $A = \{2^*, 3^*\}$ and $B = \{1^*, 3^*\}$ are not comparable, because $A \oplus B^\star = \{1, 2^*\}$ and $B \oplus A^\star = \{1^*, 2\}$.

Therefore, if we follow the definition of basic operations + and - for real numbers, we can define $B \ominus A := B \oplus A^\star$. So we obtain that

$$A \leq B \;\leftrightarrow\; \emptyset \leq B \oplus A^\star \;\leftrightarrow\; \emptyset \leq B \ominus A.$$

Finally, we present the summary of the axiomatic system of σ-Set Theory.

4.1. **The Axiom of Empty σ-set.** There exists a σ-set which has no σ-elements, that is

$$(\exists X)(\forall x)(x \notin X).$$

4.2. **The Axiom of Extensionality.** For all σ-classes $\hat{X}$ and $\hat{Y}$, if $\hat{X}$ and $\hat{Y}$ have the same σ-elements, then $\hat{X}$ and $\hat{Y}$ are equal, that is

$$(\forall \hat{X}, \hat{Y})[(\forall z)(z \in \hat{X} \leftrightarrow z \in \hat{Y}) \to \hat{X} = \hat{Y}].$$

4.3. **The Axiom of Creation of σ-Class.** We consider an atomic formula $\Phi(x)$ (where $\hat{Y}$ is not free). Then there exists the classes $\hat{Y}$ of all σ-sets that satisfies $\Phi(x)$, that is

$$(\exists \hat{Y})(x \in \hat{Y} \leftrightarrow \Phi(x)),$$

with $\Phi(x)$ a atomic formula where $\hat{Y}$ is not free.

4.4. **The Axiom of Scheme of Replacement.** The image of a σ-set under a normal functional formula Φ is a σ-set.

4.5. **The Axiom of Pairs.** For all X and Y σ-sets there exists a σ-set Z, called fusion of pairs of X and Y, that satisfy one and only one of the following conditions:

(a): Z contains exactly X and Y,
(b): Z is equal to the empty σ-set,

that is

$$(\forall X, Y)(\exists Z)(\forall a)[(a \in Z \leftrightarrow a = X \vee a = Y)\underline{\vee}(a \notin Z)].$$

4.6. **The Axiom of Weak Regularity.** For all σ-set X, for all $\lhd x, \ldots, w \rhd \in CH(X)$ we have that $X \notin \lhd x, \ldots, w \rhd$, that is

$$(\forall X)(\forall \lhd x, \ldots, w \rhd \in CH(X))(X \notin \lhd x, \ldots, w \rhd).$$

4.7. **The Axiom of non ϵ-Bounded σ-Set.** There exists a non ϵ-bounded σ-set, that is

$$(\exists X)(\exists y)[(y \in X) \wedge (\min(X) = \emptyset \vee \max(X) = \emptyset)].$$

4.8. **The Axiom of Weak Choice.** If $\hat{X}$ be a σ-class of σ-sets, then we can choose a singleton Y whose unique σ-element come from $\hat{X}$, that is

$$(\forall \hat{X})(\forall x)(x \in \hat{X} \to (\exists Y)(Y = \{x\})).$$

4.9. **The Axiom of ϵ-Linear σ-set.** There exist σ-set X such that X has the linear ϵ-root property, that is

$$(\exists X)(\exists y)(y \in X \wedge X \in LR).$$

4.10. **The Axiom of One and One* σ-set.** For all ϵ-linear singleton, there exists a ϵ-linear singleton Y such that X is totally different from Y, that is

$$(\forall X \in SG \cap LR)(\exists Y \in SG \cap LR)(X \not\equiv Y).$$

4.11. **The Axiom of Completeness (A)..** If X and Y are σ-sets, then

$$\{X\} \cup \{Y\} = \{X, Y\},$$

if and only if X and Y satisfy one of the following conditions:

- **(a):** $\min(X, Y) \neq |1 \vee 1^*| \wedge \min(X, Y) \neq |1^* \vee 1|$.
- **(b):** $\neg(X \not\equiv Y)$.
- **(c):** $(\exists w \in X)[w \notin \min(X) \wedge \neg\Psi(z, w, a, Y)]$.
- **(d):** $(\exists w \in Y)[w \notin \min(Y) \wedge \neg\Psi(z, w, a, X)]$.

4.12. **The Axiom of Completeness (B)..** If X and Y are σ-sets, then

$$\{X\} \cup \{Y\} = \emptyset,$$

if and only if X and Y satisfy the following conditions:

- **(a):** $\min(X, Y) = |1 \wedge 1^*| \vee \min(X, Y) = |1^* \wedge 1|$;
- **(b):** $X \not\equiv Y$;
- **(c):** $(\forall z)(z \in X \wedge z \notin \min(X)) \rightarrow \Psi(z, w, a, Y))$;
- **(d):** $(\forall z)(z \in Y \wedge z \notin \min(Y)) \rightarrow \Psi(z, w, a, X))$.

4.13. **The Axiom of Exclusion.** For all σ-sets X, Y, Z, if Y and Z are σ-elements of X then the fusion of pairs of Y and Z contains exactly Y and Z, that is

$$(\forall X, Y, Z)(Y, Z \in X \rightarrow \{Y\} \cup \{Z\} = \{Y, Z\}).$$

4.14. **The Axiom of Power σ-set.** For all σ-set X there exists a σ-set Y, called power of X, whose σ-elements are exactly the σ-subsets of X, that is

$$(\forall X)(\exists Y)(\forall z)(z \in Y \leftrightarrow z \subseteq X).$$

4.15. **The Axiom of Fusion.** For all σ-sets X and Y, there exists a σ-set Z, called fusion of all σ-elements of X and Y, such that Z contains σ-elements of the σ-elements of X or Y, that is

$$(\forall X, Y)(\exists Z)(\forall b)(b \in Z \rightarrow (\exists z)[(z \in X \vee z \in Y) \wedge (b \in z)]).$$

4.16. **The Axiom of Generated σ-set.** For all σ-sets X and Y there exists a σ-set, called the σ-set generated by X and Y, whose σ-elements are exactly the fusion of the σ-subsets of X with the σ-subsets of Y, that is

$$(\forall X, Y)(\exists Z)(\forall a)(a \in Z \leftrightarrow (\exists A \in 2^X)(\exists B \in 2^Y)(a = A \cup B)).$$

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